Mathematics Assessment Project

Formative Assessment Lesson Materials

Classifying Solutions to Systems of Equations

MARS Shell Center
University of Nottingham & UC Berkeley
Alpha Version

Please Note:

These materials are still at the "alpha" stage and are not expected to be perfect. The revision process concentrated on addressing any major issues that came to light during the first round of school trials of these early attempts to introduce this style of lesson to US classrooms. In many cases, there have been very substantial changes from the first drafts and new, untried, material has been added.

We suggest that you check with the Nottingham team before releasing any of this material outside of the core project team.

If you encounter errors or other issues in this version, please send details to the MAP team c/o map.feedback@mathshell.org.

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Classifying Solutions to Systems of Equations

Mathematical goals

This lesson unit is intended to help you assess how well students are able to:

- Classify solutions to a pair of linear equations by considering their graphical representations.

In particular, this unit aims to help you identify and assist students who have difficulties in:

- Using substitution to complete a table of values for a linear equation.
- Identifying a linear equation from a given table of values.
- Graphing and solving linear equations.

Common Core State Standards

This lesson involves *mathematical content* in the standards from across the grades, with emphasis on:

8.EE: Analyze and solve linear equations and pairs of simultaneous linear equations.

This lesson involves a range of *mathematical practices,* with emphasis on:

1. Make sense of problems and persevere in solving them.
6. Attend to precision.

Introduction

The lesson unit is structured in the following way:

- Before the lesson, students attempt the assessment task individually. You then review students' work and formulate questions that will help them improve their solutions.
- During the lesson, students work collaboratively in pairs or threes, plotting graphs, completing tables of values and deducing equations. Then, based on the number of their common solutions, students link these representations.
- After a final whole-class discussion, students work individually on a new the assessment task.

Materials required

- Each individual student will need two copies of the assessment task *Working with Linear Equations*, a mini-whiteboard, wipe, and pen.
- For each pair of students provide cut up copies of *Card Set A: Equations, Tables & Graphs* and *Card Set B: Arrows*, one copy of a *Graph Transparency*, copied onto a transparency, a transparency pen, a large sheet of paper for making a poster, and a glue stick.
- Provide rulers if requested.
- There are some projector resources to help with whole-class discussion.

Time needed

Fifteen minutes before the lesson for the assessment task, a seventy-minute lesson, and ten minutes in a follow up lesson (or for homework). All timings are approximate, depending on the needs of your students.

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Before the lesson

Assessment task: *Working with Linear Equations* (15 minutes)

Have the students do this task, in class or for homework, a day or more before the formative assessment lesson. This will give you an opportunity to assess the work, and identify students who have misconceptions or need other forms of help. You will then be able to target your help more effectively in the follow-up lesson.

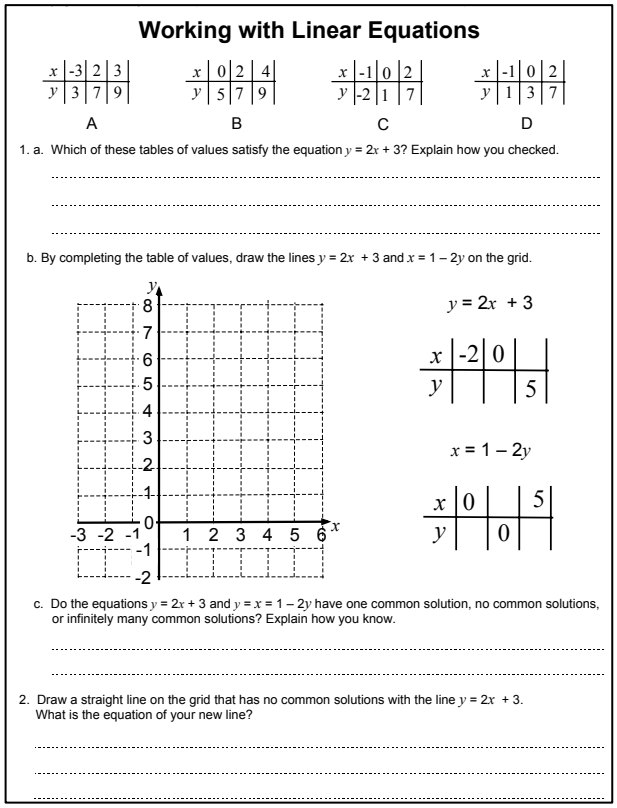
Give each student a copy of *Working with Linear Equations*.

Introduce the task briefly, and help the class to understand what they are being asked to do. You may want to explain to the class the term 'common solution'.

> *Spend fifteen minutes working individually, answering these questions.*
> *Show all your work on the sheet.*
> *Make sure you explain your answers really clearly.*

It is important that students answer the questions without assistance, as far as possible.

Students should not worry too much if they cannot understand or do everything, because you will teach a lesson using a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these. This is their goal.



Working with Linear Equations

	A	B	C	D
x	-3, 2, 3	0, 2, 4	-1, 0, 2	-1, 0, 2
y	3, 7, 9	5, 7, 9	-2, 1, 7	1, 3, 7

1. a. Which of these tables of values satisfy the equation $y = 2x + 3$? Explain how you checked.

b. By completing the table of values, draw the lines $y = 2x + 3$ and $x = 1 - 2y$ on the grid.

$y = 2x + 3$

x	-2	0	
y			5

$x = 1 - 2y$

x	0		5
y		0	

c. Do the equations $y = 2x + 3$ and $y = x = 1 - 2y$ have one common solution, no common solutions, or infinitely many common solutions? Explain how you know.

2. Draw a straight line on the grid that has no common solutions with the line $y = 2x + 3$. What is the equation of your new line?

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and difficulties. The purpose of this is to forewarn you of the issues that will arise during the lesson, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this is counterproductive, as it encourages students to compare scores, and distracts their attention from how they may improve their mathematics.

Instead, help students to make further progress by asking questions that focus attention on aspects of their work. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest that you write your own lists of questions, based on your own students' work, using the ideas in the *Common issues* table. Preferably, write questions on each student's work, but if you do not have time for this, then prepare a few questions that apply to most students and write these on the board when the assessment task is revisited.

Common issues:	Suggested questions and prompts:
Student assumes that only one of the tables satisfies the equation $y = 2x + 3$ (Q1) For example: The student selects only table A.	• *Are there more than three pairs of values that satisfy the equation $y = 2x + 3$?* • *Have you checked the values for x and y in each of the tables?*
Student makes an incorrect assumption about the multiplicative properties of zero (Q1 & Q2) For example: The student assumes $2 \times 0 + 3 = 5$. They then may select Table B as satisfying the equation $y = 2x + 3$ (Q1.)	• *Is 4×0 the same as 4×1?* • *Use addition to figure out two multiplied by zero. [E.g. $0 + 0 = 0$.]*
Student applies the rules for multiplying negative numbers incorrectly (Q1 & Q2) For example: The student assumes $2 \times -1 + 3 = 5$ They then may select Table C as satisfying the equation $y = 2x + 3$ (Q1.)	• *Is 3×-2 the same as 3×2?*
Student provides little or no explanation (Q1)	• *What method did you use when checking which tables satisfy the equation? Write what you did.*
Student incorrectly draws the graph For example: The student draws a non-linear graph.	• *On your graph, is the slope always the same? Does this agree with the equation of the graph?* • *How can you check you have plotted the graph correctly?*
Student uses guess and check to complete the tables of values (Q1b)	• *Can you think of a quicker method?* • *Would changing the subject of the equation help you figure out some of the values?*
Student states that the two equations, $y = 2x + 3$ and $x = 1 - 2y$ have no common solutions (Q1c) For example: The student fails to extend the line $x = 1 - 2y$ beyond the values in the table. This means the two lines do not intersect.	• *What does 'common solution' mean?* • *Are there any other points that satisfy the equation $x = 1 - 2y$? Plot some.*
Student provides little or no explanation (1c and 2)	• *Explain why you think your answer is correct.*
Student either does not plot a line that has no common solutions with the line $y = 2x + 3$ or plots it incorrectly (Q2)	• *Sketch two lines that have no common solutions. What property do they share?[The lines will be parallel.]*
Student uses guess and check to figure out the equation of the line (Q2)	• *Can you think of a quicker method?* • *What can you tell me about two lines with no common solution? Give me two equations that have no common solution.*

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Suggested lesson outline

Whole-class introduction (10 minutes)

Give each student a mini-whiteboard, pen, and wipe. Maximize participation in the discussion by asking all students to show you solutions on their mini-whiteboards.

Write the equation $y = 3x + 2$ on the board.

Ask the following questions in turn:

If $x = 5$ what does y equal? [17]

If $x = -1$ what does y equal? [-1]

Ask students to explain how they arrived at their answer. If a variety of values are given within the class, discuss any common mistakes and explore different strategies.

Remind students of the rules for multiplying negative numbers and emphasize the need to take care when calculating with directed numbers. The misconception that x and y must always have different values as they are distinct letters can also be addressed here.

If $y = 8$ what does x equal? [2]

If $y = 0$ what does x equal? [-⅔]

Students may either use guess and check, or rearrange the equation in order to figure out the value for x. You may want to discuss these two strategies.

Students often think that they have made a mistake when they get an answer that is not a whole number. Discuss the value of checking an answer by substituting it back in as x, as well as emphasizing that not all solutions will be positive integers and that negative and fractional solutions can also occur.

It may also be appropriate to discuss the benefits of leaving answers in fraction form, rather than converting to a decimal, especially when a recurring decimal will result. Provide an example of say, ½, and discuss the difference between this fraction expressed as a decimal, and -⅔ expressed as a decimal.

How can you check your answer?

How can you check that all your answers are correct? [Sketch the coordinates on a grid and see if they form a straight line.]

Explain to students that in the next activity they will be using their skills of substitution and of solving equations to help them to investigate graphical representations of linear equations.

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Collaborative activity: *Completing Card Set A: Equations, Tables & Graphs* (20 minutes)

Organize students into pairs.

For each pair provide a cut up copy of *Card Set A: Equations, Tables & Graphs*.

> *These six cards each include a linear equation, a table of values and a graph. However, some of the information is missing.*
>
> *In your pairs, complete each card so that it has an equation, a completed table of values and a graph. Make sure that the graph fills the grid in the same way as it does on Card C1. Take care to plot points carefully.*
>
> *When one card is complete, explain it to your partner. Ask your partner to check your card. If your partners' answers are different, ask for an explanation.*
>
> *You may need to do some calculations to complete the cards. Use your mini-whiteboard for this.*
>
> *It is important that you both understand and agree on the answers.*

Slide 1 of the projector resource summarizes these instructions.

It does not matter if students are unable to complete all six cards. It is more important that they can confidently explain their strategies and have a thorough understanding of the skills they are using.

While students work in small groups you have two tasks, to make a note of students approaches to the task, and to support student reasoning.

Make a note of student approaches to the task

Listen and watch students carefully. Note any common mistakes. Also notice the way in which students complete the cards.

> Do students use the completed table of values to plot the graph or do they use their knowledge of slope and intercept to draw the graph directly from the equation?
> Do students first plot the line using easy values for x or y, and then read off values from the graph in order to complete the table?
> Do students rearrange the equation or do they use guess and check to solve for x or y?
> Do students use multiplication to eliminate the fraction from the equation?
> Do students use the slope and intercept or guess and check to figure out the equation of the graph?

You will be able to use this information in the whole class discussion.

Support student reasoning

You may want to use the questions in the *Common issues* table to help address misconceptions.

Try not to make suggestions that move students towards a particular strategy. Instead, ask questions to help students to reason together.

> *Martha completed this card. Jordan, can you explain Martha's work?*
> *Show me a different method from your partner's to check their method is correct.*

If you find the student is unable to answer this question, ask them to discuss the work further. Explain that you will return in a few minutes to ask a similar question.

> *How can you check the card is correct? [Read off coordinates from the line, use the slope and intercept to check the equation matches the line, etc.]*

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For each card, encourage students to explain their reasoning and methods carefully.

How do you know that $y = 3$ when $x = 2$ in Card C2? What method did you use?

How did you find the missing equation on Card C1/C3? Show me a different method.

Suppose you multiply out the equation on Card C4. What information can you then deduce about the graph? [The y-intercept and slope.]

Which of these equations is arranged in a way that makes it easy to draw a graph using information about the lines' y-intercept and slope? [C5.] What are they?

You may find some students struggle when the slope of a line is negative or when dealing with negative signs or when the slope is a fraction.

If students are struggling, suggest that they focus on Cards C4 and C5 first.

Checking work (10 minutes)

Ask students to exchange completed cards with another pair of students.

Carefully check their cards.

Point out to the other pair any cards you think are incorrect.

Make any necessary changes to your own cards.

Collaborative activity: Using *Card Set B* to link *Card Set A* (20 minutes)

As students finish checking cards:

I would like you to spend a few minutes comparing your completed cards. Select two cards and note on your whiteboards any common properties of the equations or/and the graphs. Repeat this for all of your completed cards.

Give each pair *Card Set B: Arrows*, one copy of *Graph Transparency*, a transparency pen, a large sheet of paper for making a poster, and a glue stick.

Stick the six cards on your poster paper with a gap between each. Don't worry if some of them still need completing as you may have some time to complete these later.

You are going to link your completed cards with an arrow card.

Each pair of equations will either have no common solutions, one common solution or infinitely many common solutions.

For pairs of cards that have one common solution, you need to complete the arrow card with the values of x and y where this solution occurs.

It may help if you copied the lines from your cards onto the transparency.

Use the information on your whiteboards about the common properties of the equations or/and graphs.

Slide 2 of the projector resource displays these instructions.

Modeling drawing a line on the transparency and overlaying it on a paper copy of a card may help students understand the role of the transparency in this activity.

As students work on this, support them as before. Again you may want to use some of the questions in the *Common issues*. Walk around and ask students to explain their decisions.

Again notice how students are working. Once students have recognized that there is one common solution, are they checking the solution algebraically as well as using the graphs? Are any students relying purely on the algebraic representation of the equation?

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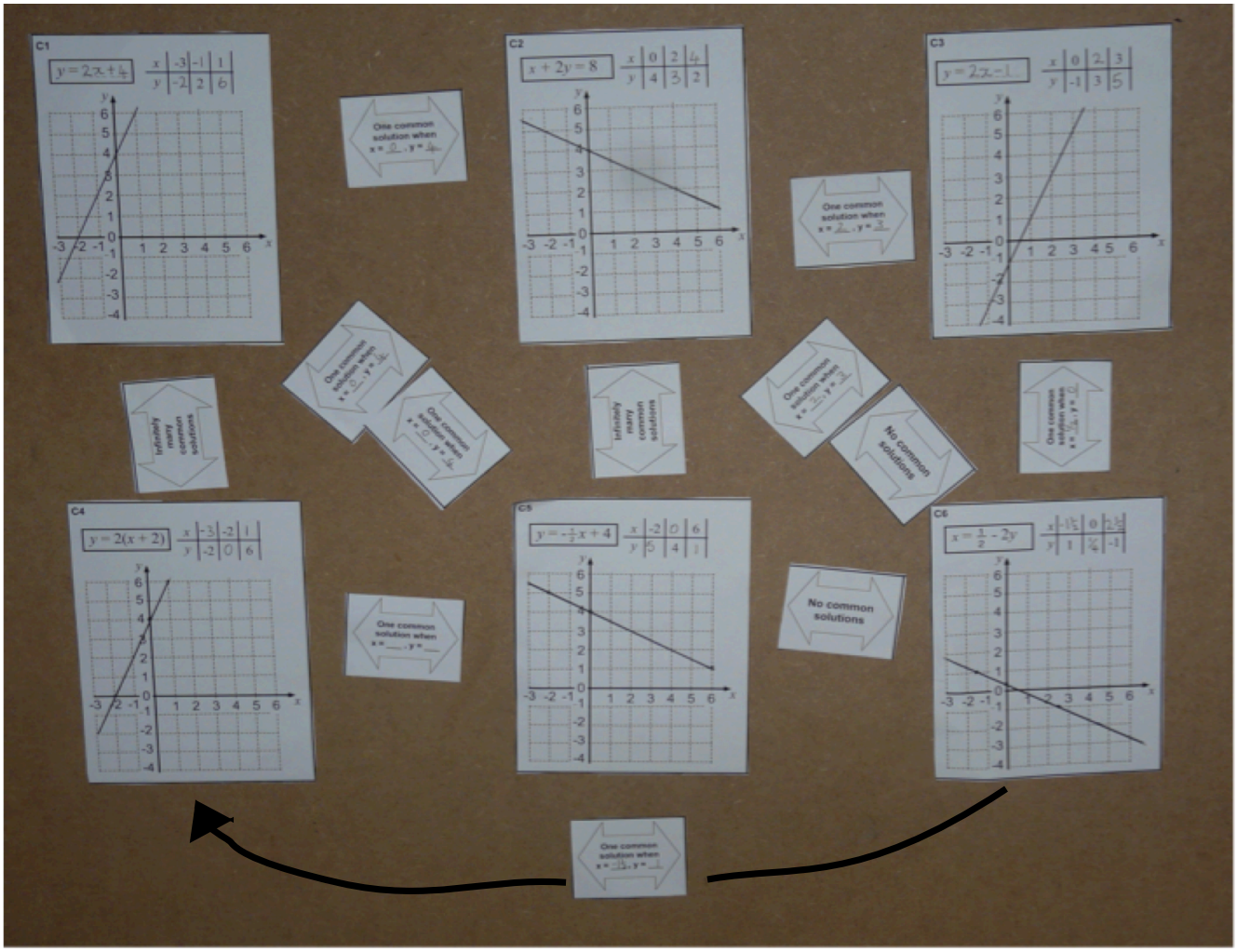
Once students have linked all completed cards, they can go back and finish any uncompleted cards from *Card Set A* and link these as well. There will be a total of eleven links when all six graphs have been investigated.

The finished poster produced may look something like this:



Whole-class discussion (10 minutes)

Organize a discussion about what has been learned. The intention is that you focus on students understanding of the different strategies, not just checking that everyone produced the same poster. Use what you have noticed in the collaborative activities to inform your questions.

Again use the mini-whiteboards to encourage all students to participate in the discussion.

Ask the following questions in turn:

1. *Show me two equations that have one common solution. [E.g. $y = 2x + 4$ and $y = -\frac{1}{2}x + 4$].*

   What are the solution values for x and y? [$x = 0$ and $y = 4$.]

   What happens to the graphs at this point? [They intersect each other.]

   Show me two more equations that have one common solution. This time use equations that do not appear on the cards. Sketch their graphs.

2. *Show me two equations that have no common solutions. [$y = 2x + 4$ and $y = 2x – 1$.]*

   How do you know they have no common solution? [They are parallel lines so will never intersect.]

   What do you notice about these two equations? [They have the same coefficient of x/same slope.]

   Show me two more equations that have no common solutions. This time use equations that do not appear on the cards. Sketch their graphs.

3. *Give me two equations with infinitely many common solutions. [E.g. $y =2x + 4$ and $y = 2(x + 2)$.]*

   What do you notice about the two graphs for these equations? [They are the same line.]

   Why is this? [$2(x + 2)$ is $2x + 4$ in factorized form.]

The focus of this discussion is to explore the link between the graphical representations of the equations and their common solutions, even though students may well have used both the algebraic representation and the table of values during the classification process. Allow students to describe their methods to the rest of the class, whilst helping them to recognize that solutions to a system of two linear equations in two variables correspond to the points of intersection of their graphs as well as what it means graphically, when there are no or infinitely many common solutions.

Improving individual solutions to the assessment task (10 minutes)

Return to the students their response to the original assessment task, as well as a copy of the task *Working With Linear Equations (Revisited.)*

Ask students to look again at their original individual solutions to the problems, together with your comments.

Read through your original solutions to the Working With Linear Equations problems.

Make some notes on what you have learned during the lesson.

Now have a go at the second sheet: Working With Linear Equations (Revisited). Can you use what you have learned to answer these questions?

If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only the questions they think are appropriate to their own work. If you are short of time, then you could set this task in the next lesson or for homework.

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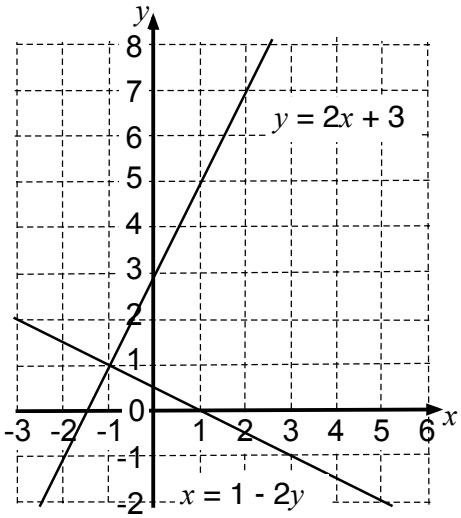
Solutions

Assessment task: *Working with Linear Equations*

1. a. Tables A and D satisfy the equation $y = 2x + 3$.

   Table B satisfies the equation $y = x + 5$ and table C is non-linear.

   b.



$y = 2x + 3$

x	-2	0	1
y	-1	3	5

$x = 1 - 2y$

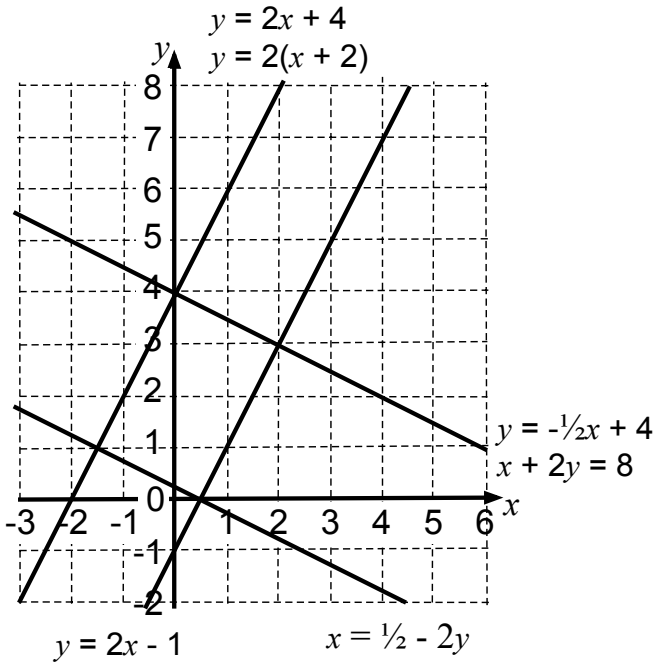
x	0	1	5
y	0.5	0	-2

   c. The two graphs have one common solution at $x = -1$, $y = 1$. This is the point of intersection of the two graphs.

2. Students can draw any line that has the same slope as $y = 2x + 3$. For example $y = 2x$ or $y = 2x + 1$ etc.

Lesson task: *Card Sets A and B*

The six cards in *Card Set A* describe the four straight lines below:



Card 1

$y = 2x + 4$

x	–3	–1	1
y	–2	2	6

Card 2

$x + 2y = 8$

x	0	2	4
y	4	3	2

Card 3

$y = 2x – 1$

x	0	2	3
y	–1	3	5

Card 4

$y = 2(x + 2)$

x	–3	–2	1
y	–2	0	6

Card 5

$y = - ½x + 4$

x	–2	0	6
y	5	4	1

Card 6

$x = ½ – 2y$

x	–1.5	0	2.5
y	1	0.25	–1

Infinitely many common solutions

$x + 2y = 8$ and $y = – ½x + 4$ $x + 2y = 8$ (C2) is a rearrangement of $y = –½x + 4$.

$y = 2(x + 2)$ and $y = 2x + 4$ $y = 2(x + 2)$ (C4) is the factorized form of $y = 2x + 4$.

No common solutions

$y = 2x + 4$ and $y = 2x – 1$ Equal slopes.

$y = –½x + 4$ and $x = ½ – 2y$ Equal slopes.

Equal slopes.

One common solution

$y = 2x + 4$ (or $y = 2(x + 2)$) and $y = –½x + 4$ (or $x + 2y = 8$) have one common solution at (0,4).

$y = 2x + 4$ (or $y = 2(x + 2)$) and $x = ½ – 2y$ have one common solution at (–1.5,1).

$y = 2x – 1$ and $y = –½x + 4$ (or $x + 2y = 8$) have one common solution at (2,3).

$y = 2x – 1$ and $x = ½ – 2y$ have one common solution at (0.5,0).

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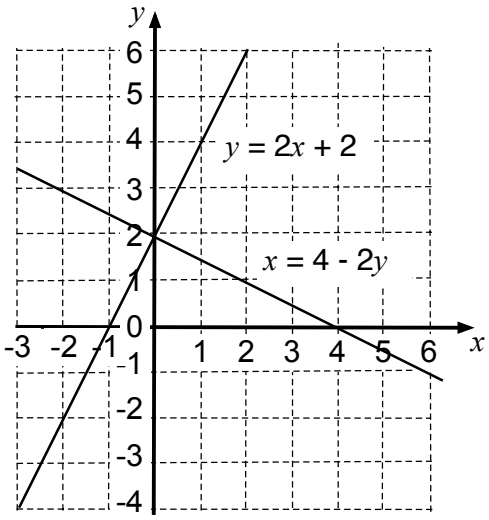
Assessment task: *Working with Linear Equations (Revisited)*

1. a. Tables B and D satisfy the equation $y = 2x + 2$.

   Table A is non-linear and table C satisfies the equation $y = 3x + 1$.

   b.



$y = 2x + 2$

x	–3	0	2
y	–4	2	6

$x = 4 - 2y$

x	2	4	6
y	1	0	–1

   c. The two graphs have one common solution at $x = 0$, $y = 2$. This is the point of intersection of the two graphs.

2. Students can draw any line that has the same slope as $y = 2x + 2$. For example $y = 2x$ or $y = 2x + 1$ etc.

Working with Linear Equations

x	-3	2	3
y	-3	7	9

A

x	0	2	4
y	5	7	9

B

x	-1	0	2
y	5	1	7

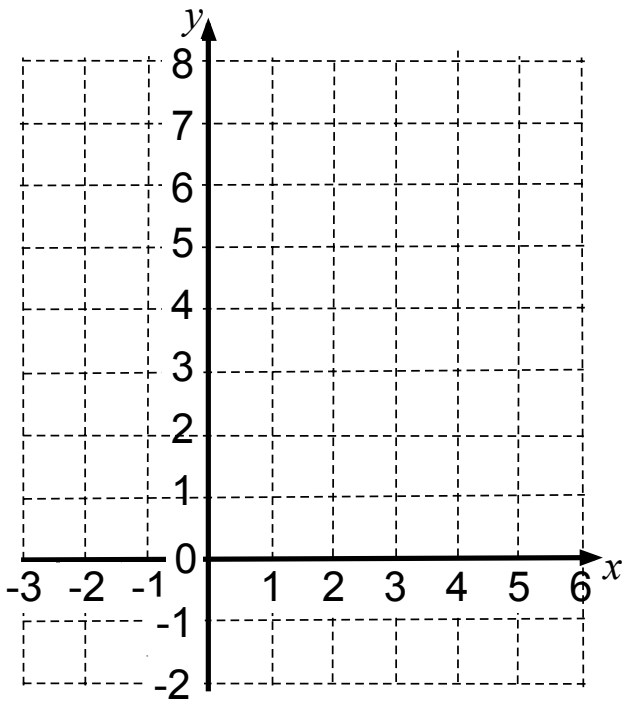
C

x	-1	0	2
y	1	3	7

D

1. a. Which of these tables of values satisfy the equation $y = 2x + 3$? Explain how you checked.

b. By completing the table of values, draw the lines $y = 2x + 3$ and $x = 1 - 2y$ on the grid.



$y = 2x + 3$

x	-2	0	
y			5

$x = 1 - 2y$

x	0		5
y		0	

c. Do the equations $y = 2x + 3$ and $y = x = 1 - 2y$ have one common solution, no common solutions, or infinitely many common solutions? Explain how you know.

2. Draw a straight line on the grid that has no common solutions with the line $y = 2x + 3$. What is the equation of your new line? Explain your answer.

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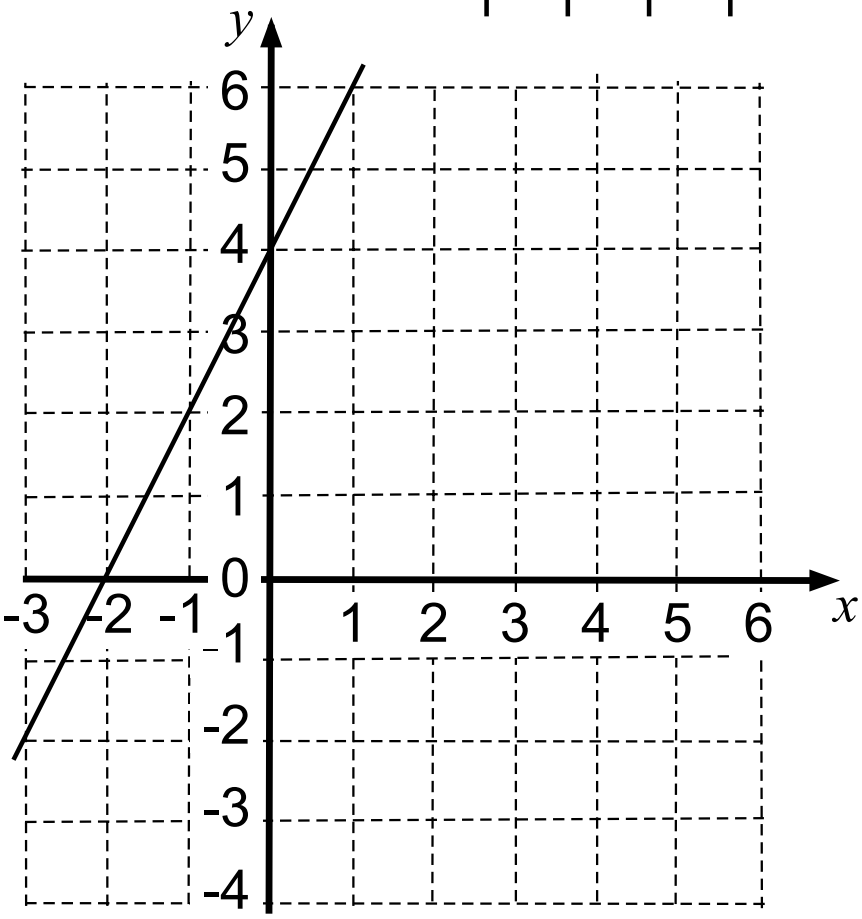
Card Set A: Equations, Tables & Graphs

C1

$y =$ _____

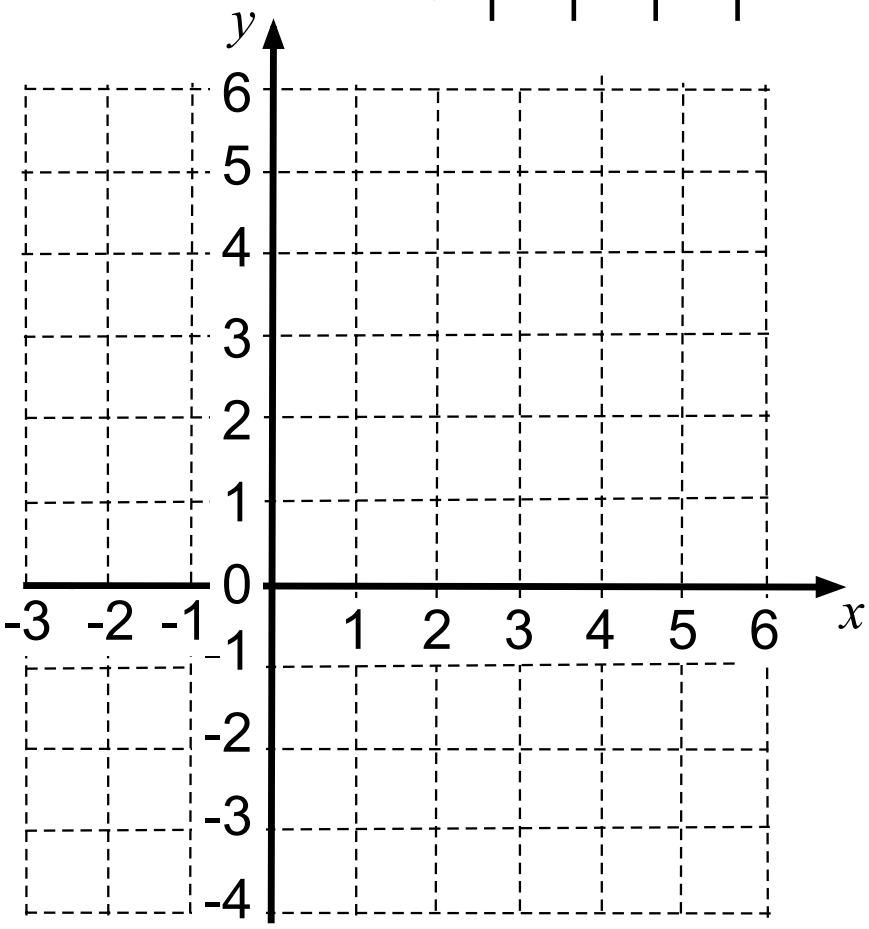
x	-3		1
y		2	



C2

$x + 2y = 8$

x	0	2	
y	4		2



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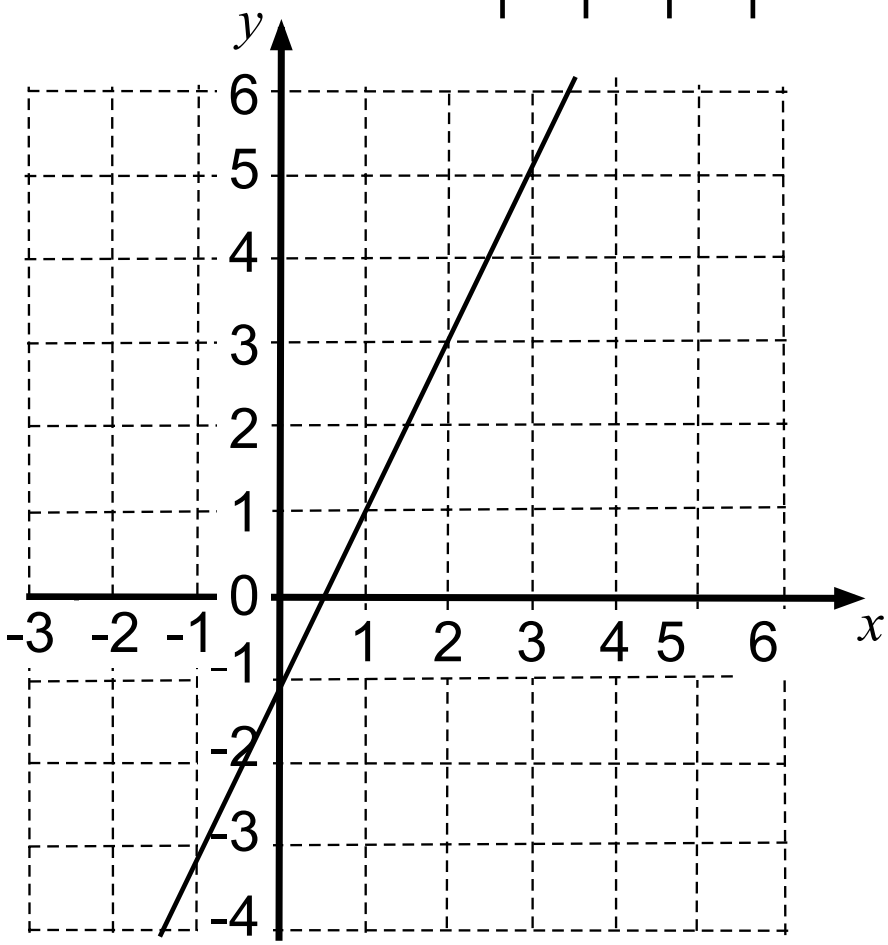
Card Set A: Equations, Tables & Graphs (continued)

C3

$y =$ _____

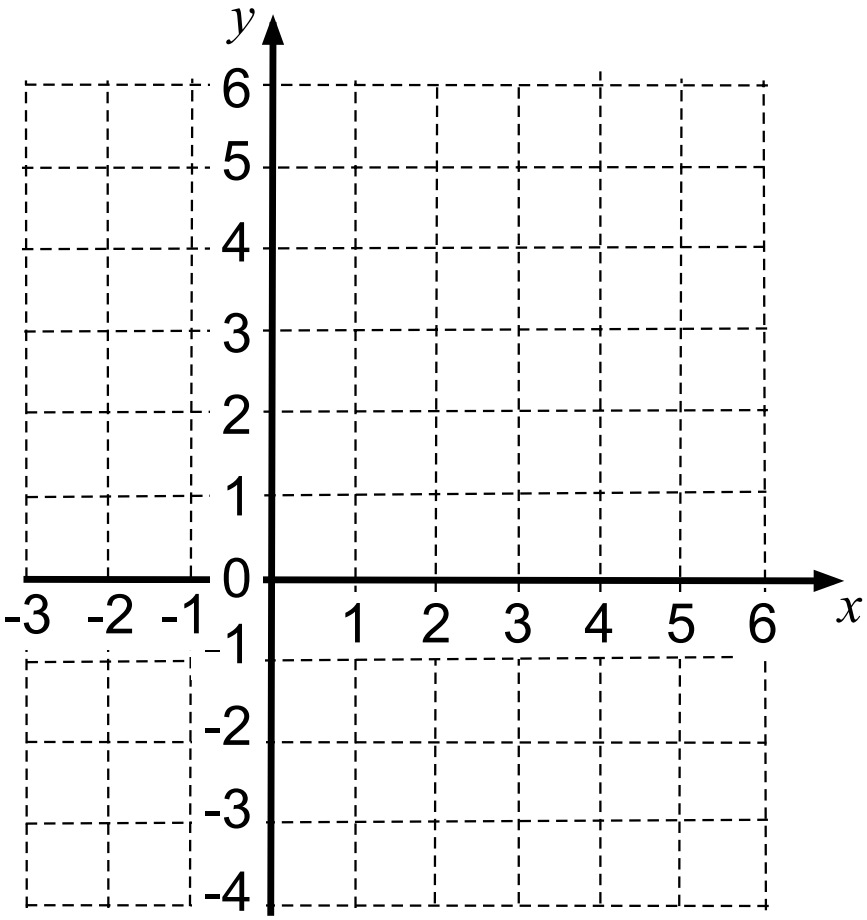
x	0		3
y	-1	3	



C4

$y = 2(x + 2)$

x		-2	
y	-2		6



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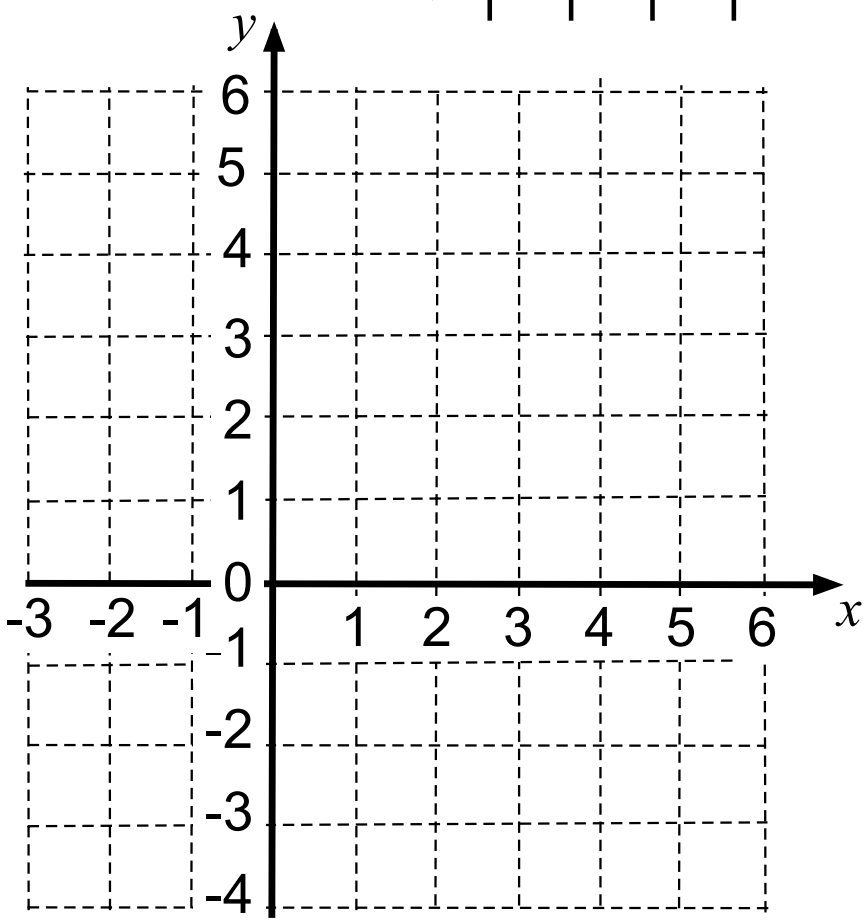
Card Set A: Equations, Tables & Graphs (continued)

C5

$$y = -\tfrac{1}{2}x + 4$$

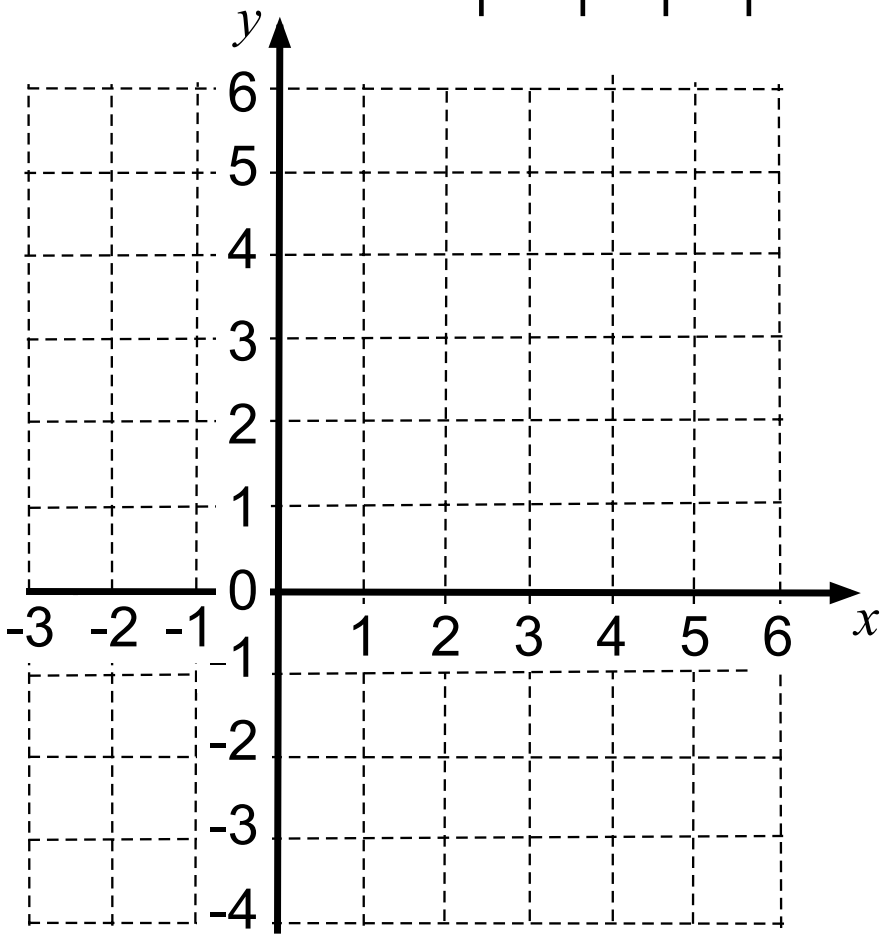
x	-2		6
y		4	



C6

$$x = \frac{1}{2} - 2y$$

x		0	
y	1		-1



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Card Set B: Arrows

No common solutions	No common solutions	Infinitely many common solutions	Infinitely many common solutions
One common solution when x = ___ , y = ___	One common solution when x = ___ , y = ___	One common solution when x = ___ , y = ___	One common solution when x = ___ , y = ___
One common solution when x = ___ , y = ___	One common solution when x = ___ , y = ___	One common solution when x = ___ , y = ___	One common solution when x = ___ , y = ___

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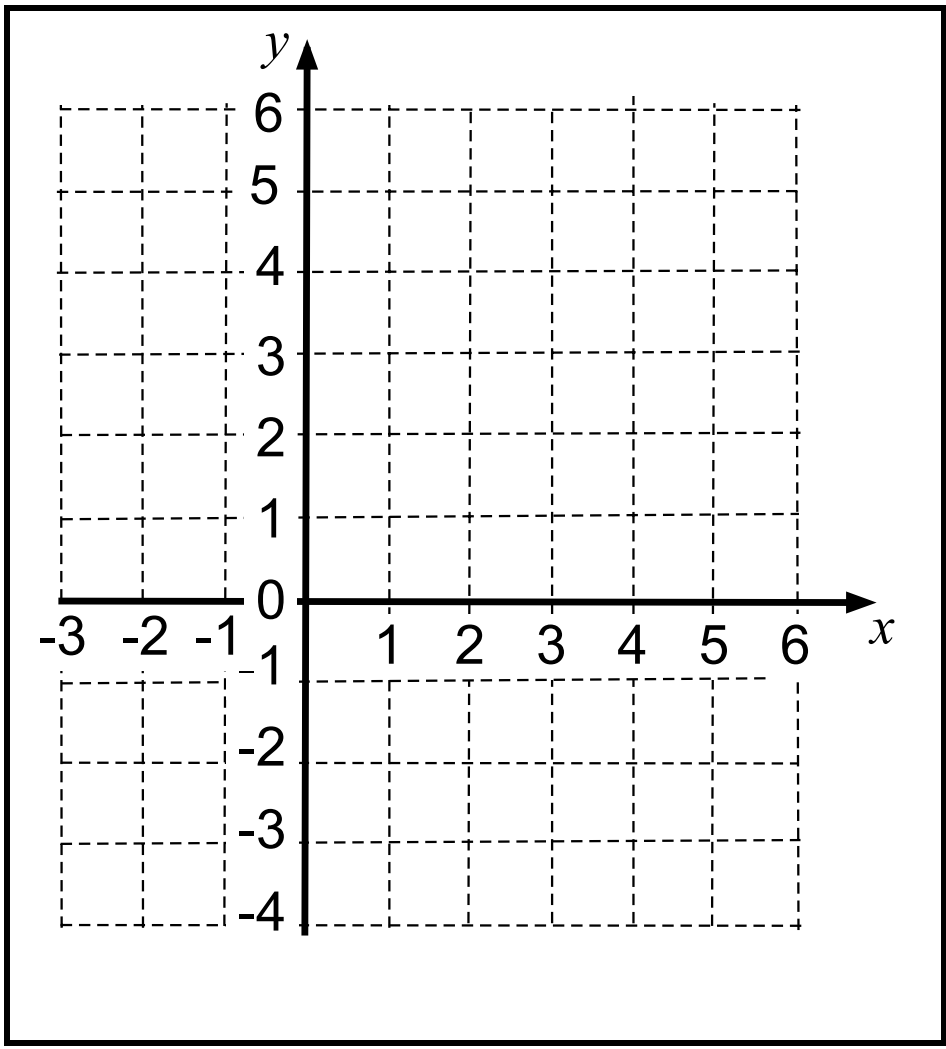
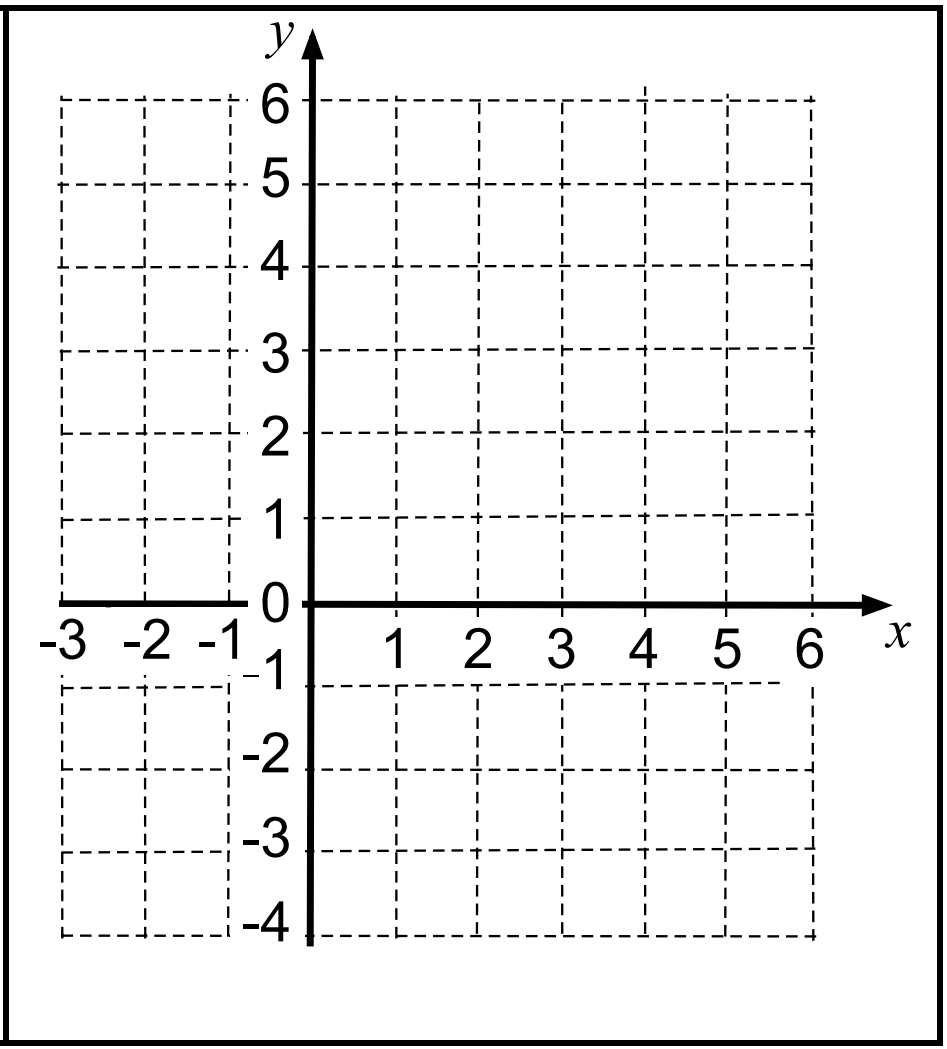
Graph Transparency





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Working with Linear Equations (Revisited)

x	0	1	2
y	4	4	5

A

x	1	2	3
y	4	6	8

B

x	-2	-1	1
y	-5	-2	4

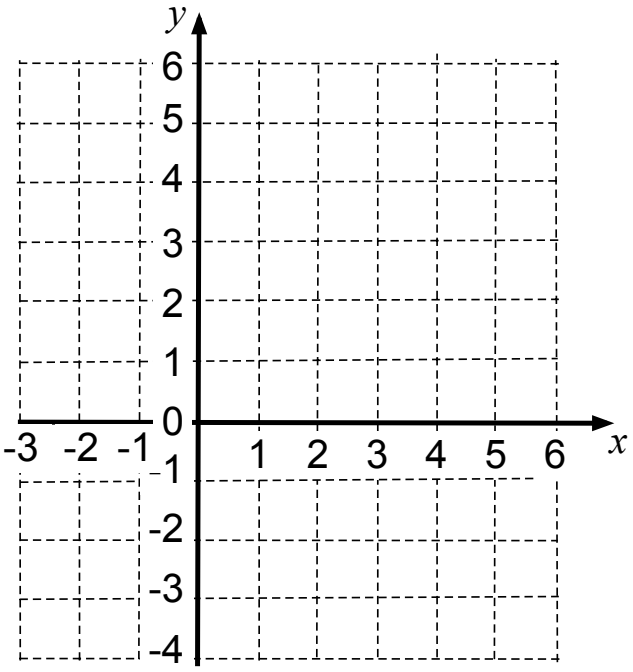
C

x	-2	-1	1
y	-2	0	4

D

1. a. Which of these tables of values satisfy the equation $y = 2x + 2$? Explain how you checked.

b. By completing the table of values, draw the lines $y = 2x + 2$ and $x = 4 - 2y$ on the grid.



$y = 2x + 2$

x	-3	0	
y			6

$x = 4 - 2y$

x		4	6
y	1		

c. Do the equations $y = 2x + 2$ and $x = 4 - 2y$ have one common solution, no common solutions, or infinitely many common solutions? Explain how you know.

2. Draw a straight line on the grid that has no common solutions with the line $y = 2x + 2$. What is the equation of your new line?

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Working Together 1

1. In pairs, complete each card so that it has an equation, a completed table of values and a graph.
   - Make sure that the graph fills the grid in the same way as it does on Card C1.
   - Take care to plot points carefully.

2. When one card is complete:
   - Explain it to your partner.
   - Ask your partner to check your card.
   - If your partners answers are different, ask for an explanation.

3. You may need to do some calculations to complete the cards. Use your mini-whiteboard for this.

It is important that you both understand and agree on all the answers.

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Working Together 2

1. You are going to link your completed cards with an arrow card.

2. Each pair of equations will either have no common solutions, one common solution or infinitely many common solutions. Looking at the graphs carefully may help you to decide.

3. For pairs of cards that have one common solution, you need to complete the arrow card with the values of x and y where this solution occurs.

4. It may help if you copied the lines from your cards onto the transparency.

It is important that you both understand and agree on all the answers.

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