Mathematics Assessment Project

Formative Assessment Lesson Materials

Estimating:
Counting Trees

MARS Shell Center
University of Nottingham & UC Berkeley
Alpha Version

If you encounter errors or other issues in this version, please send details to the MAP team c/o map.feedback@mathshell.org.

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Estimating: *Counting Trees*

Mathematical goals

This lesson unit is intended to help you assess how well students are able to:

- Solve simple problems involving ratio and direct proportion.
- Choose an appropriate sampling method.
- Collect discrete data and record them using a frequency table.

Common Core State Standards

This lesson involves a range of *mathematical practices* from the standards, with emphasis on:

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.

This lesson also asks students to select and apply mathematical content from across the grades, including the *content standards*:

7.RP: Analyze proportional relationships and use them to solve real-world and mathematical problems.
7.SP: Use random sampling to draw inferences about a population.

Introduction

The unit is structured in the following way:

- Before the lesson, students attempt the *Counting Trees* task individually. You then review their work, and create questions for students to answer in order to improve their solutions.
- At the start of the lesson, students again work individually on *Counting Trees* task, answering your questions.
- Next, students work collaboratively in small groups. Their task is to produce a better solution to the *Counting Trees* task, than they did individually. Then, working in the same small groups, students analyze responses to the *Counting Trees* task, written by students in another class.
- In a whole-class discussion, students compare and evaluate the methods they have seen and used.
- In the final part of the lesson, students spend ten minutes reviewing what they have learned.

Materials required

- Each individual student will need a copies of the task *Counting Trees, The Farm,* and the questionnaire *How Did You Work?*
- Each small group of students will need a new copy of the task, *Counting Trees*, and *The Farm,* a sheet of poster paper, pen, and copies of all three copies of *Sample Responses to Discuss*.
- There are some projector resources to help introduce activities and support whole-class discussions.

Time needed

Approximately fifteen minutes before the lesson, a seventy minute lesson and ten minutes in a follow-up lesson (or for homework.) Timings given are only approximate. Exact timings will depend on the needs of your class.

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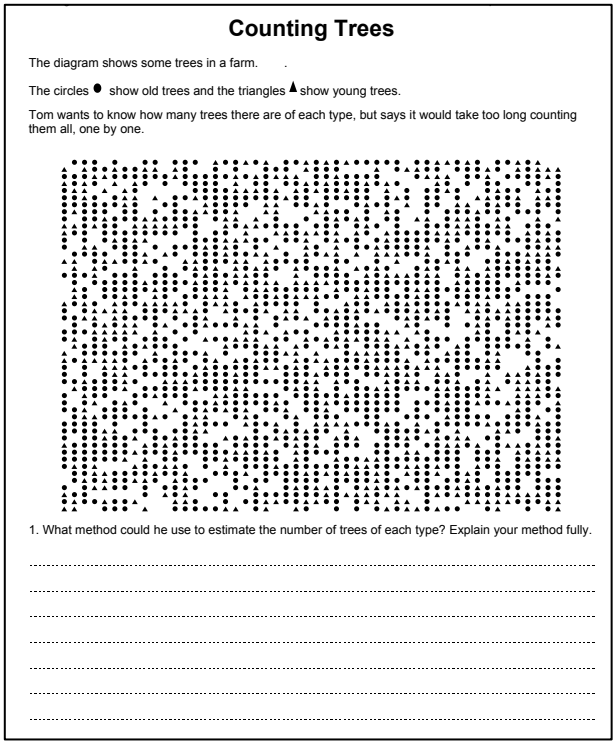
Before the lesson

Assessment task: *Counting Trees* (15 minutes)

Have the students do this task, in class or for homework, a day or more before the lesson. This will give you an opportunity to assess their work, and to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.



Counting Trees

The diagram shows some trees in a farm.

The circles ● show old trees and the triangles ▲ show young trees.

Tom wants to know how many trees there are of each type, but says it would take too long counting them all, one by one.

1. What method could he use to estimate the number of trees of each type? Explain your method fully.

Does anyone know what a tree farm is?

How is a farm different from a natural forest?

Do you think there could be areas in the farm where there are very few trees? What could be the reason for this? [Area has poor soil, less light, trees have been felled etc.]

Give each student a copy of *Counting Trees* and *The Farm.* Briefly introduce the task.

Now explain what you are asking students to do.

Spend fifteen minutes on your own, reading through the questions and trying to answer them as carefully as you can.

The sheet, The Farm, is an extra copy of the farm on the Counting Trees sheet. You may want to use the sheet to try out ideas.

Don't worry if you can't do everything. There will be a lesson on this material that will help you improve your work. Your goal is to be able to answer these questions with confidence by the end of that lesson.

You may want to show the class Slide 1 of the projector resource.

It is important that students are allowed to answer the questions without assistance, as far as possible. If students are struggling to get started then ask questions that help them understand what is required, but make sure you do not do the task for them. You may need to remind students that Tom does not want to count all the trees in the farm.

Students who sit together often produce similar answers, and then when they come to compare their work, they have little to discuss. For this reason we suggest that, when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

When all students have made a reasonable attempt at the task, tell them that they will have time to revisit and revise their solutions later.

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Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their problem solving strategies.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare scores, and distract their attention from what they can do to improve their mathematics.

Instead, help students make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given the *Common Issues* table. We suggest that you make a list of your own questions, based on your students' work, using the ideas below. You may choose to write questions on each student's work. If you do not have time to do this, select up to five questions that will help to the majority of students. These can be written on the board at the beginning of the lesson.

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Common issues:	Suggested questions and prompts:
Student chooses a method which does not involve any sampling For example: The student counts the trees. *Or*: The student multiplies the number of columns by the number of rows, and then to take account of gaps, halves this answer.	• *Read the question again. Have you done what is asked?* • *What assumptions have you made? Are your assumptions reasonable?* • *How could you improve your estimate?*
Student chooses a sampling method that is unrepresentative For example: The student counts the trees in the first row/column and multiples by the number of rows/columns. *Or*: The student multiplies the number of trees in the left column by the number of trees in the bottom row.	• *How could you improve/check your estimate?* • *Which rows/columns have you left out of your calculations?*
Student uses area and perimeter in their calculations	• *What does the area measure?* • *What does the perimeter measure?*
Student makes incorrect assumptions For example: The student does not account for gaps. *Or*: The student does not realize that there are an unequal number of trees of each kind.	• *Is there a pattern to how the trees are distributed in the farm? Does your work assume there is a pattern?* • *What does your method assume? Is this a reasonable assumption?*
Student only calculates the number of trees in one area of the farm	• *How do you know your sample area is a good representation of the whole farm?* • *How could you check the accuracy of your estimate?*
Students' work is difficult to follow	• *Would someone unfamiliar with this type of work understand your work?*
Student chooses an appropriate sampling method	• *Can you suggest a second, different sampling method?* • *If you miscount your sample by 1, how does that affect your overall estimate?*
Student completes the task	• *If I now gave you a very large jar of the same sized, colored sweets, how could you estimate the fraction, which are red?*

87

88

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Suggested lesson outline

Introduction: *Counting Trees* (10 minutes)

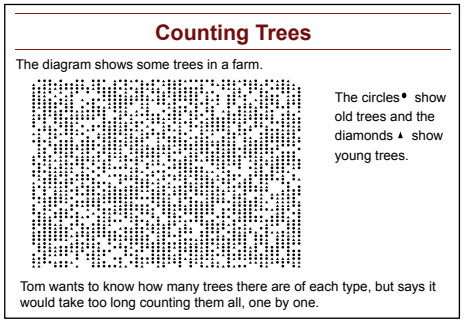
Return your students' work on *Counting Trees*.

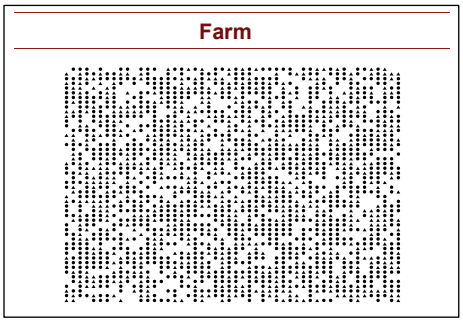
If you have not added questions to individual pieces of work, then write your list of up to say five questions on the board. Students are to select questions from the board that are appropriate to their own work, and spend a few minutes answering them. If students have difficulty selecting questions then they are to answer all of them. Doing this should help them recognize whether the question does apply to their work.

Begin the lesson by briefly reintroducing the problem:

> *Recall what we were working on previously. What was the task?*
>
> *I have had a look at your work, and I have some questions I would like you to think about.*
>
> *Carefully read through your own work and the questions I have written.*
>
> *Answer only the questions that relate to your own work.*
>
> *At this point you do not need to do the math.*

You may want to show the class Slide 1 or/and Slide 2 of the projector resource.





This is an opportunity for students to review their work. Whilst they are doing this listen and watch students carefully. Are students able to reflect on their own work and suggest useful improvements? Ask questions that help students to clarify their thinking.

Collaborative activity: *Improving the Solution* (20 minutes)

When students have answered the questions, organize them into groups of two or three.

Give each group a piece of poster paper, a copy of *The Farm,* a glue stick, and a pen.

Planning a Joint Solution

Ask students to glue the sheet *The Farm* to the middle of the poster.

Ask students to share their ideas about the task, and plan a joint solution.

> *I want you to share your work with your group.*
>
> *Take it in turns to explain how you did the task and how you now think it could be improved.*
>
> *Listen carefully to any explanation. Ask questions if you don't understand or agree with the method. [You may want to use some of the questions I have written on the board.]*
>
> *I want you to plan a joint approach that is better than your separate solutions.*

Ask students to write their plan on their poster, above the sheet *The Farm.*

Slide 3 of the projector resource summarizes these instructions.

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Implementing the Plan

Students are now to show their joint solution on the poster. The aim is not to produce a poster with a perfect solution, but one that shows how students have collaboratively developed their ideas.

While students work individually you have two tasks, to note their different approaches to the task, and to support student reasoning.

Note different student approaches to the task

Listen and watch students carefully. Note the different student approaches to the task and what assumptions they make. Do students choose an appropriate sampling method? Do they check their estimate by considering a different sample? What makes them decide they have looked at enough sample areas?

Support student problem solving

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking.

> *Why did you select this row? Do you think the number of trees in this row will be different from the number in this one? Why do you think this is the case?*
>
> *How can you check you have a good estimate?*
>
> *Have you made any assumptions? Do you think your assumption(s) are reasonable?*
>
> *What is the point of counting the trees in more than one sample?*
>
> *How do you decide how many sample areas to use? [If there is a big variation in the number of trees then students may want to count the number of trees in a third area.]*
>
> *How do you decide where to take your samples?*
>
> *What is the difference between an estimate and a guess?*

You may want to use some of the questions in the *Common Issues* table to support your own questioning.

If the whole class is struggling on the same issue, write relevant questions on the board and hold an interim discussion.

Collaborative analysis of *Sample Responses to Discuss* (20 minutes)

When students have had sufficient time to attempt the problem in their group, give each group copies of the *Sample Responses to Discuss*. This task gives students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy.

> *Here are some different solutions to the problem. Imagine you are the teacher.*
>
> *Describe how the student approached the problem.*
>
> *What do you like/dislike about the math?*
>
> *Has the student made any assumptions?*
>
> *How could the student improve the solution?*

Slide 4 of the projector resource summarizes these instructions.

Encourage students to focus on the math of the student work, not whether the student has neat writing etc.

There may not be time for all groups to look at all three solutions, and it is not essential that they do. It is more important that students provide a detailed evaluation of the work they are critiquing and suggest ways that it could be improved.

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Laura attempts to estimate the number of old and new trees by multiplying the number along each side of the whole diagram and then halving. She does not account for gaps nor does she realize that there are an unequal number of trees of each kind.

Can you explain why Laura halves her answer? What assumption is she making?

① You could multiply the number of trees in the length by the number of trees in the width and then half your answer.

② a. Old trees - 644
Young trees - 644
width - 33
length - 39.
33x39= 1287
1287÷ 2= 643.5-644

Woody uses a sample of two columns and counts the number of old and young trees. He then multiplies by 25 (half of 50 columns) to find an estimate of the total number. He does, however, take account of the different numbers of old and new trees.

Why do you think Woody has rounded his answers?

2 columns has 21 young
55 old trees

50 columns is approx
50 ÷ 2 = 25
25 x 21 = amount of young trees = 525
25 x 55 = amount of old trees = 1,375
rounded up
young 530
old 1,380

Amber chooses a representative sample and carries through her work to get a reasonable answer. She correctly uses proportional reasoning. She checks her work as she goes along by counting the gaps in the trees. Her work is clear and easy to follow, although a bit inefficient.

Can you explain why Amber multiplies by 25 in her method?

Counting trees

1. If Tom draws a 10X10 square round some trees and counts how many old and new there are. There are 50 rows and 50 columns altogether so he must multiply by 25. He could do this a few times to check and then take the average.

2.
53 old x 25 = 1325 old
28 new x 25 = 700 new
19 spaces x 25 = 475 spaces
100 2500

1325+1200÷2=1262.5
700+875 ÷2= 787.5

Check
48 old x 25 = 1200 old
35 new x 25 = 875 new
17 spaces x 25 = 425 spaces
100 2500

So about 1263 old trees and 788 new trees

Plenary whole-class discussion: comparing different solution methods (20 minutes)

Organize a whole-class discussion comparing the three given solutions. Collect comments and ask for explanations. You may want to use Slides 5, 6, and 7 to support this discussion.

Ask students to evaluate and compare responses.

Which one did you like best? Why?

Which approach did you find most difficult to understand? Why?

Did anyone come up with a method different from/similar to these?

Again you may want to use some of the questions in the *Common issues* table to support your own questioning.

Review solutions to *Counting Trees* (10 minutes)

Give the questionnaire *How You Work* to each student.

Ask students to and complete the review questionnaire. Some teachers set this task as homework.

The questionnaire may help student monitor and review their progress during and at the end of an activity.

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Solutions

There are many ways of completing this task, but solutions should include the following:

Students should describe an appropriate sampling technique which takes into account the different proportions of old and new trees as well as allowing for the gaps in between.

- Students should check their figures for the number of old and new trees by counting the number of trees in at least two sample areas.
- If there is a big variation in the number of trees in these two areas then students may want to count the number of trees a third sample area.
- Each sample should cover a distinctly different area of the farm (that is two sample areas should not be next to each other.)
- Each sample should cover the same sized area.
- Students may then calculate the average number of old and new trees for these sample areas.
- These averages should then be multiplied up to obtain an estimate for the total number of old and new trees.

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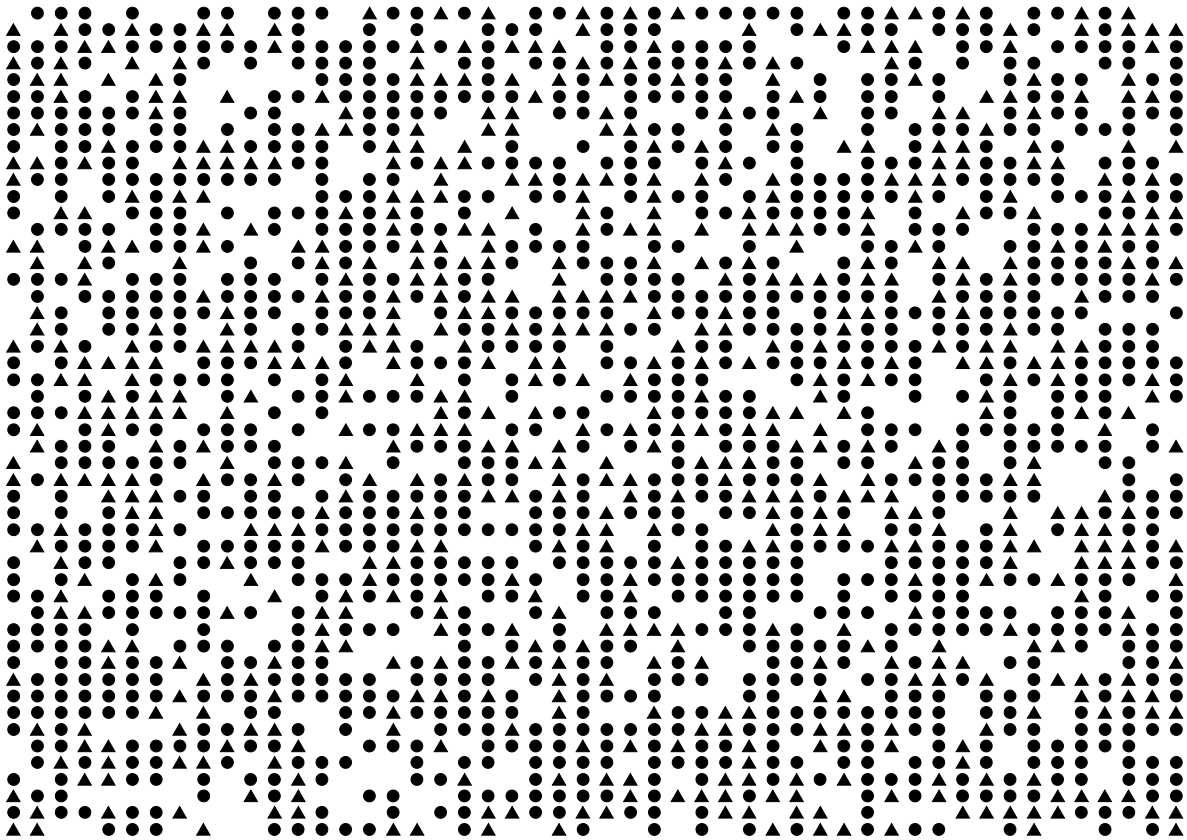
Counting Trees

The diagram shows some trees in a farm.

The circles ● show old trees and the triangles ▲ show young trees.

Tom wants to know how many trees there are of each type, but says it would take too long counting them all, one by one.



1. What method could Tom use to estimate the number of trees of each type?

   Explain your method fully.

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2. Use your method to estimate the number of:

(a) Old trees.

(b) New trees.

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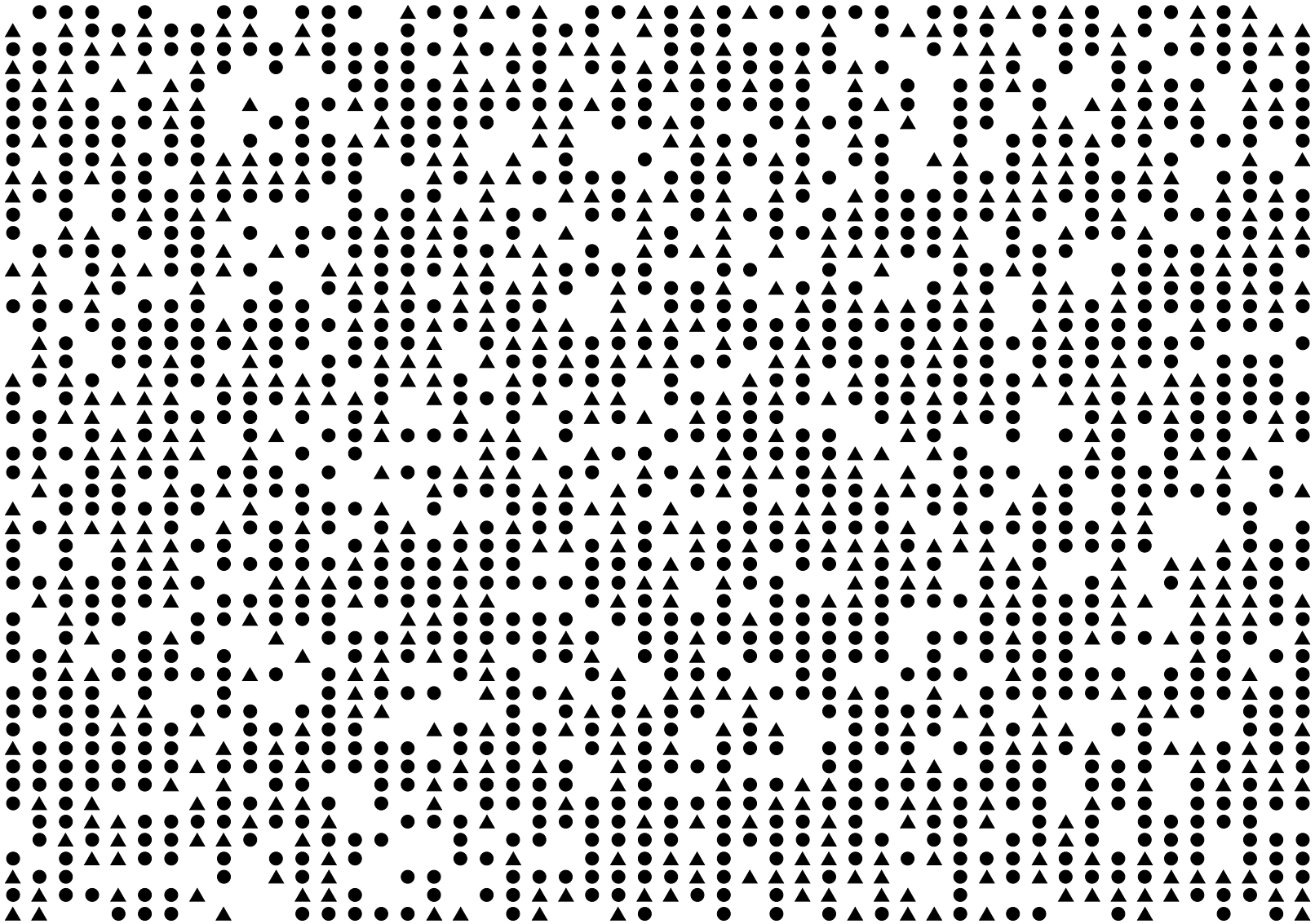
The Farm



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Sample Responses to Discuss: Laura

Imagine you are Laura's teacher. Describe how Laura approached the problem.

- What do you like/dislike about the work?
- Has Laura made any assumptions?
- How could Laura improve her solution?

① You could multiply the number of trees in the length by the number of trees in the width and then of half your answer.

② a. Old trees - 644
Young trees - 644

width - 33
length - 39.

33x39 = 1287
1287 ÷ 2 = 643.5 - 644

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Sample Responses to Discuss: Woody

Imagine you are Woody's teacher. Describe how Woody approached the problem.

- What do you like/dislike about the work?
- Has Woody made any assumptions?
- How could Woody improve her solution?

2 columns has 21 young trees
55 old

50 columns is approx
50 ÷ 2 = 25
25 × 21 = amount of young trees = 525
25 × 55 = amount of old trees = 1,375
rounded up
young 530
old 1,380

Sample Responses to Discuss: Amber

Imagine you are Amber's teacher. Describe how Amber approached the problem.

- What do you like/dislike about the work?
- Has Amber made any assumptions?
- How could Amber improve her solution?

Counting trees

1. If Tom draws a 10X10 square round some trees and counts how many old and new there are. There are 50 rows and 50 columns altogether so he must multiply by 25. He could do this a few times to check and then take the average.

2.

53 old	x 25	=	1325	old
28 new	x 25	=	700	new
19 spaces	x 25	=	475	spaces
100			2500	

1325 + 1200 ÷ 2 = 1262.5
700 + 875 ÷ 2 = 787.5

Check

48 old	x 25	=	1200	old
35 new	x 25	=	875	new
17 spaces	x 25	=	425	spaces
100			2500	

So about 1263 old trees and 788 new trees

How Did You Work?

Tick the boxes and complete the sentences that apply to your work.

1. Our group work was better than my own work ☐

   Our joint solution was better because .....

   .....

   .....

   .....

2. Why did you decide on the size and positiion of your sample? .....

   .....

   .....

3. We decided to count the number of trees in just one place ☐ We decided to count the number of trees in more than one place ☐

   We made this decision because .....

   .....

   .....

   .....

4. We made some assumptions ☐

   These assumptions were .....

   .....

   .....

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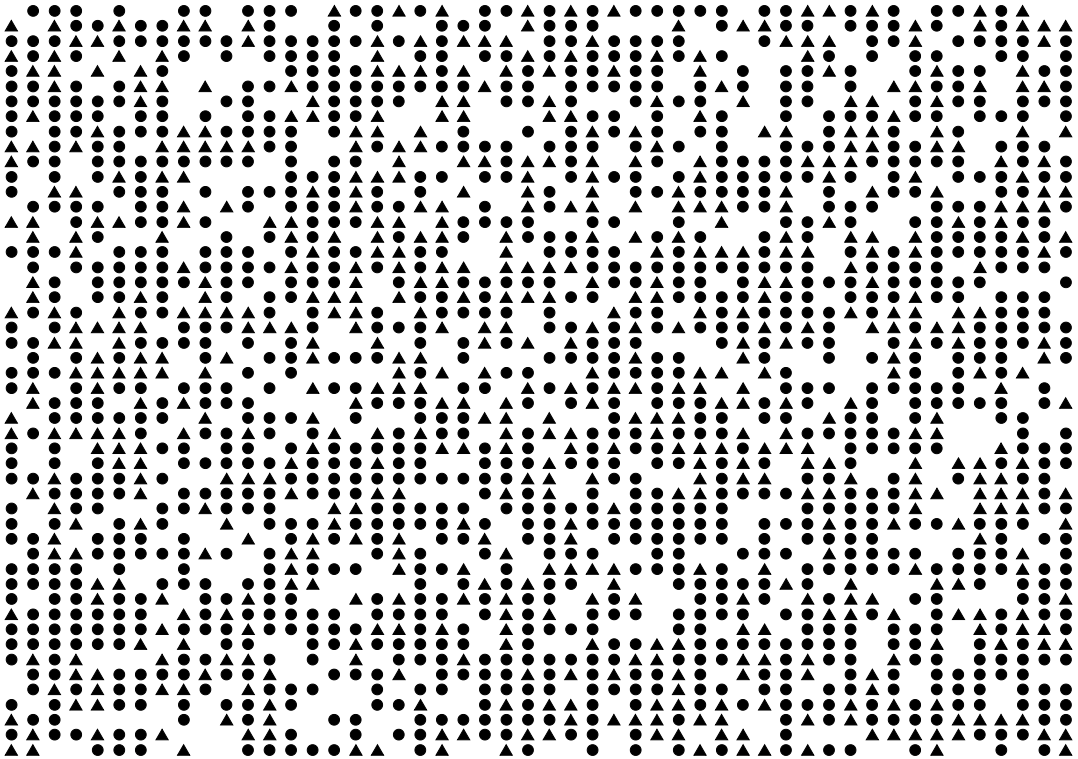
Counting Trees

The diagram shows some trees in a farm.



The circles ● show old trees and the diamonds ▲ show young trees.

Tom wants to know how many trees there are of each type, but says it would take too long counting them all, one by one.

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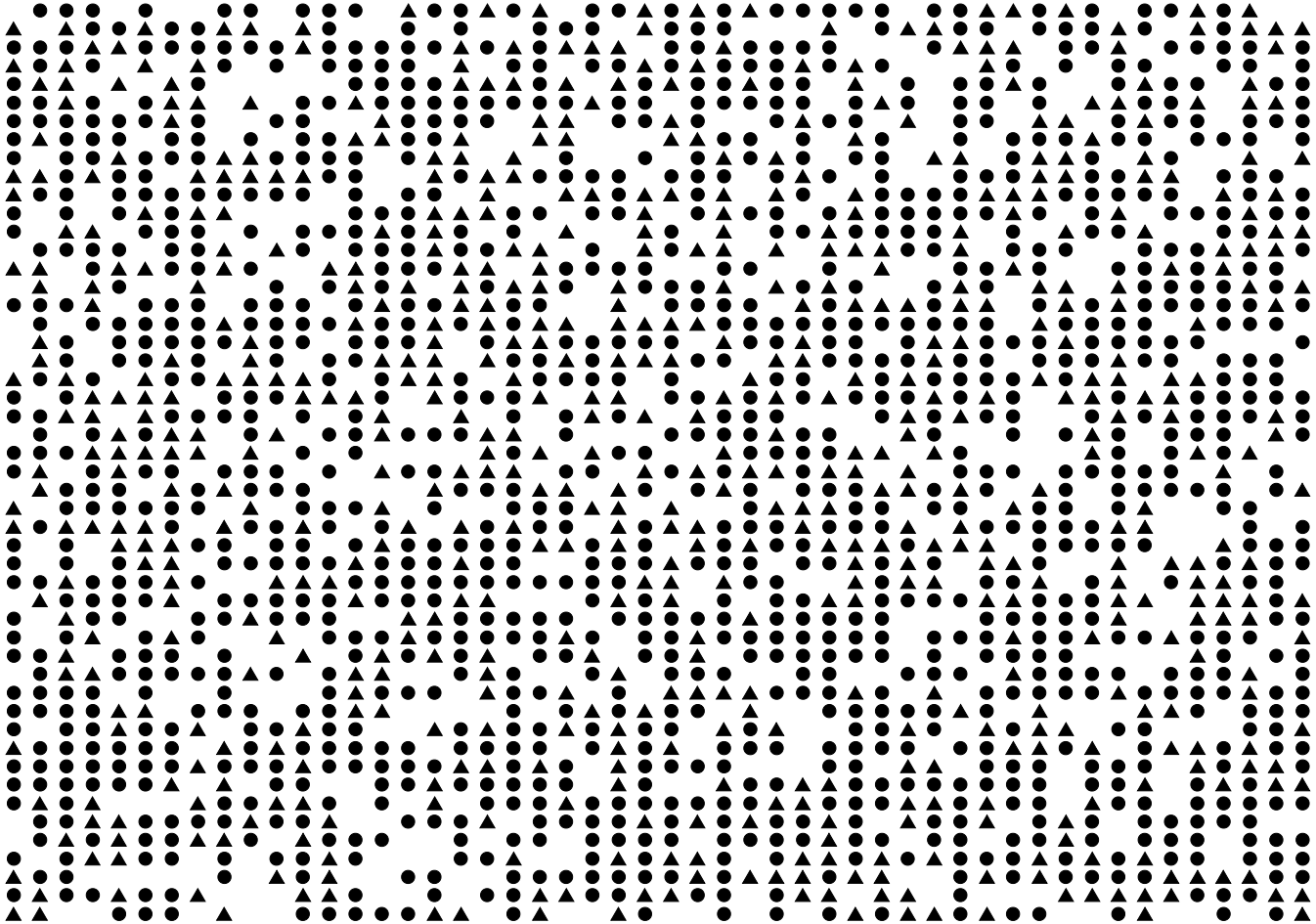
Farm



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Planning a Joint Solution

1. Take it in turns to explain how you did the task and how you now think it could be improved.

2. Listen carefully to explanations.
   - Ask questions if you don't understand.
   - Discuss with your partners:
     - What you like/dislike about your partners' math.
     - Any assumptions your partner has made.
     - How their work could be improved.

3. Once everyone in the group has explained their solution, plan a joint approach that is better than each of the separate solutions.

4. Write an outline of your plan on your poster, above the handout *The Farm.*

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Analyzing Sample Responses to Discuss

Explain what the student has done.

- What do you like/dislike about the math?
- Has the student made any assumptions?
- How could the solution be improved?

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Laura's method

① You could multiply the number of trees in the length by the number of trees in the width and then half your answer.

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Young trees - 644

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Woody's method

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55 old

50 columns is approx
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25 × 21 = amount of young trees = 525
25 × 55 = amount of old trees = 1,375
rounded up

young 530
old 1,380

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Amber's method

Counting trees

1. If Tom draws a 10X10 square round some trees and counts how many old and new there are. There are 50 rows and 50 columns altogether so he must multiply by 25. He could do this a few times to check and then take the average.

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