Mathematics Assessment Project

Formative Assessment Lesson Materials

The Pythagorean Theorem: *Square Areas*

MARS Shell Center
University of Nottingham & UC Berkeley
Alpha Version

Please Note:

These materials are still at the "alpha" stage and are not expected to be perfect. The revision process concentrated on addressing any major issues that came to light during the first round of school trials of these early attempts to introduce this style of lesson to US classrooms. In many cases, there have been very substantial changes from the first drafts and new, untried, material has been added.

We suggest that you check with the Nottingham team before releasing any of this material outside of the core project team.

If you encounter errors or other issues in this version, please send details to the MAP team c/o map.feedback@mathshell.org.

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The Pythagorean Theorem: *Square Areas*

Mathematical goals

This lesson unit is intended to help you assess how well students are able to:

- Use the area of right triangles to deduce the areas of other shapes.
- Use dissection methods for finding areas.
- Organize an investigation systematically, and collect data.
- Deduce a generalizable method for finding lengths and areas (The Pythagorean Theorem.)

Common Core State Standards

This lesson asks students to select and apply mathematical content from across the grades, including aspects of the following *content standards*:

8.G: Understand and apply the Pythagorean Theorem.

This lesson involves a range of mathematical practices from the standards, with emphasis on:

1. Make sense of problems and persevere in solving them.
6. Attend to precision.
7. Look for and make use of structure.
8 Look for and express regularity in repeated reasoning.

Introduction

The lesson is structured in the following way:

- Before the lesson, students attempt the *Square Areas* task individually. You then review their work, and create questions for students to answer in order to improve their solutions.
- A whole-class introduction is used to pose the problem of finding the areas of "tilted squares" drawn on a square grid. Students work for 10 minutes on this problem.
- Students share different approaches for calculating areas and are offered three generalizable methods that they might use.
- Students are now asked to find all the possible areas "tilted squares" at a specific tilt. This requires a systematic approach.
- In a whole class discussion, results are shared and organized. The method is now generalized into the Pythagorean theorem.
- Finally, students work individually on a new assessment task.

Materials required

- Each student will need a copy of the task sheets *Square Areas*, *Tilted Squares*, *Proving the Pythagorean Theorem*, and *Square Areas (revisited)*, multiple copies of the *Dotted Grid Paper* (on demand).
- Each small group of students will need copies of the sheet, *Some Different Approaches*.
- There are projector resources to help introduce activities and support whole-class discussion.

Time needed

Approximately twenty minutes before the lesson, a seventy minute lesson, and ten minutes in a follow-up lesson (or for homework.) Timings given are only approximate. Exact timings will depend on the needs of your class.

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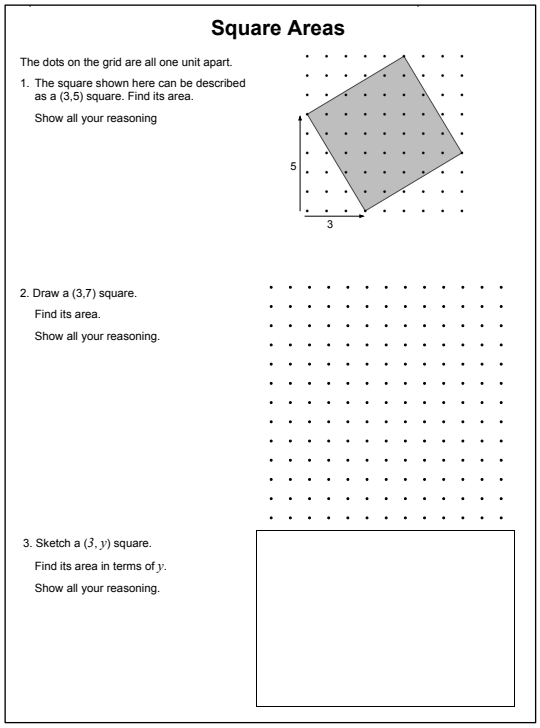
Before the lesson

Assessment task: *Square Areas* (20 minutes)


Square Areas

The dots on the grid are all one unit apart.

1. The square shown here can be described as a (3,5) square. Find its area.

   Show all your reasoning

2. Draw a (3,7) square.

   Find its area.

   Show all your reasoning.

3. Sketch a $(3, y)$ square.

   Find its area in terms of y.

   Show all your reasoning.

Have the students do this task, in class or for homework, a day or more before the lesson. This will give you an opportunity to assess their work, and to find out the kinds of difficulties students have. You will then be able to target your help more effectively in the follow-up lesson.

If you think your students will have a problem using dot paper spend time discussing with your students how to measure lengths and areas on the paper. Use a data projector to project a grid.

Each student will need a ruler, a pencil and a sheet of squared dot paper for their rough work.

Now explain what you are asking students to do.

> *Spend twenty minutes on your own, answering these questions as carefully as you can. Show all your work so that I can understand your reasoning. There will be a lesson on this material [tomorrow] that will help you improve your work.*

Project the dotted squares on the board and check that students understand the notation (*x,y*) for describing the squares perhaps by asking them to a couple of example.

> *Can you draw me a (2,1) square?*
>
> *Can someone draw me a (3,2) square?*

From now on, it is important that students are allowed to answer the questions without assistance, as far as possible. If students are struggling to get started then ask questions that help them understand what is required, but make sure you do not do the task for them.

When all students have made a reasonable attempt at the task, tell them that they will have time to revisit and revise their solutions later.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their problem solving strategies.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare scores, and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table below. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will be of help to the majority of students. These can be written on the board when this work is returned to students (see page 7).

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Common issues:	Suggested questions and prompts:
Student estimates the area of the square (Q1 and Q2) For example: The student divides the first diagram into squares and attempts to count them. Or: The student uses a ruler to measure the length of a side. OR: The student determines there is a 4 by 4 square in the center of the shaded square. Then adds to this area an estimate for the areas of the remaining four triangles (Q1.)	• *Do you think your method will give an exact answer? Why?* • *Can you think of a method that will give you a more precise answer?* • *Can you find a way of calculating the area without counting squares?* • *Dots are one unit apart. Can you find the area of the square in square units?*
Student dissects the square into smaller shapes (e.g. triangles), but these do not permit an accurate calculation (Q1 and Q2)	• *Can you find a method of dividing the square into smaller triangles, for which you know the base and the height of each triangle?* • *Can you find a method where you don't have to divide the square up into smaller pieces?*
Student uses Pythagorean theorem to figure out the areas	• *Can you find a precise way of calculating the area without using the Pythagorean theorem?*
Student has difficulty with algebraic notation (Q3)	• *In question 1, what is the value of y?* • *Can you use your method from question 1 to calculate the area?*
Student completes the task successfully	• *What other areas can you make by drawing tilted squares on the dotted grid?* • *What areas are impossible to make?* *Can you show that these are impossible?*

82

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Suggested lesson outline

Interactive Introduction: *Tilted Squares* (10 minutes)

Give students the sheet *Tilted Squares*.

> *Today we are going to do some more questions like the one I gave you yesterday. This time, you are going to work in pairs sharing your ideas.*

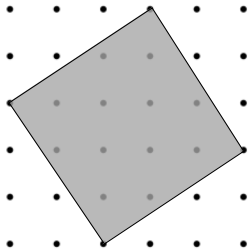
Allow students a few minutes to solve the first question individually, and then they should discuss the question with a partner.

As they do this, go around the class and look at their methods for finding the areas of the shapes. Use your findings from the pre-assessment to help you find students who may have different approaches.

Whole class discussion to share students' own approaches (5 minutes)

Use the data projector to project a grid and draw a large version of the tilted (2,3) square on the board.

> *Have any of you got a good method for finding the area?*
> *Come out and show me on the board.*



Ask students to share their methods with the whole class. Choose pairs of students that you know have different approaches. If possible, include one that involves counting squares and one that involves dissection. Focus on getting students to understand these approaches fully, before criticizing or refining them.

> *Can you explain in your own words how [Marla] approached the task?*

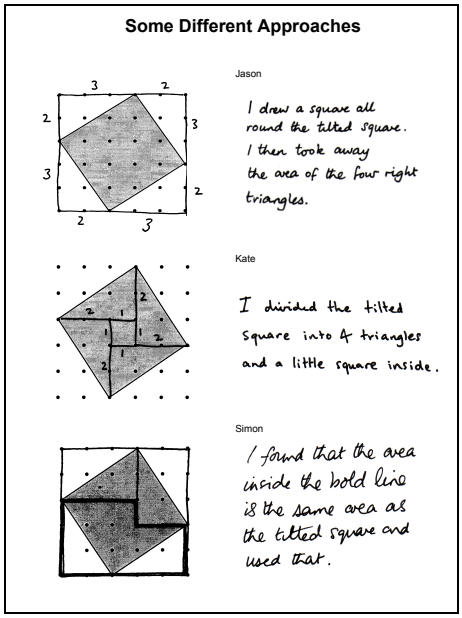
Then, each time, ask:

> *Do you think that this method will give an accurate answer? Why?*
> *Do you think this method is quick and easy to use? Why?*

Whole class discussion on the sample student work (5 minutes)

Give out the sheet: *Some different approaches*. Discuss the methods used by Jason, Kate, and Simon, projecting their diagrams on the board.

> *These might be helpful approaches.*
> *Can you use these methods to find the areas?*
>
> *Do you think they will give an accurate answer?*
> *Why or why not?*
>
> *Do you agree with Simon's statement that the area inside the bold line is the same as the area of the tilted square?*



Jason's method leads to:

Area of square – area of four triangles $= (2+3)^2 - 4 \times \frac{1}{2} \times 2 \times 3 = 13$

Kate's dissection leads to:

Area of four triangles + square in the middle $= 4 \times \frac{1}{2} \times 3 \times 2 + (3-2)^2 = 13 =$

Simon's method leads directly to: Area of tilted square = $3^2 + 2^2 = 13$

Working in pairs: Finding possible areas (15 minutes)

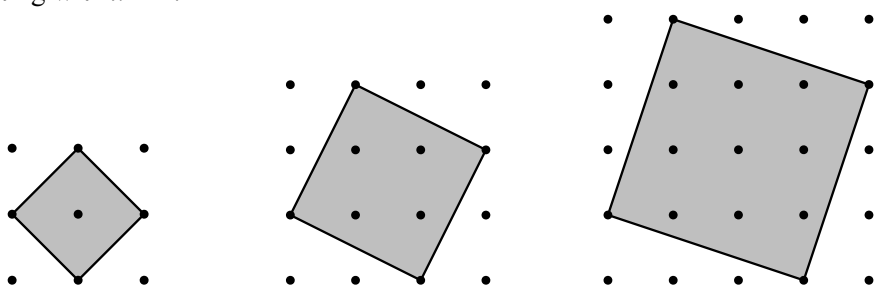
Introduce the task:

> *In your pairs, you are now going to see if there is a quick way to figure out the areas of tilted squares.*
> *Each group is to investigate the area of squares with a fixed y-value.*
> *See if you can come up with a method that you can use for all your squares.*

In trials it was found that many students were unable to organize a systematic approach for themselves. Therefore we suggest that you ask each pair of students' to work on squares with a specific y-value.

For example, one pair may work with squares with a y-value 0, another pair, squares with a y-value of 1, and so on, up to squares of a y-value of 4. It is likely that you will have several pairs working on squares with the same y-value. If a group finish quickly, they should start on a different y value.

For example, in the diagrams below, show squares of a y-value of 1, y is fixed at 1 unit and x is increased by 1 each time, starting with $x = 1$.



Have extra copies of the *Dotted Grid Paper* to give students.

While students work, note their different approaches to the task, and support their reasoning. Listen and watch students carefully. Use the questions in the *Common issues* table to help address difficulties. Prompt students with questions like:

> *Try keeping one corner of the square fixed, and move a second corner one unit along each time.*
> *How can you find the areas?*
> *Can you use the same method for calculating areas each time?*
> *Can you see a pattern in your results?*

For example, when $y = 1$, as in the diagram above, students may notice that the sequence of areas forms a pattern: 1, 2, 5, 10, 17, 26, …

> *What do you notice about this sequence of numbers? [Each is one greater than a square number.]*

If students just notice the sequence of areas increase by 1, 3, 5, 7, 9 etc. encourage them to investigate how each square area relates to the x and y values of the square.

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Whole-class discussion: Organizing and generalizing class results (15 minutes)

Draw on the board a blank table with x going from 0 to 4 and y going from 0 to 4. Leave some space to the right of the table for comments and observations.

x / y	0	1	2	3	4	Comments
0						
1						
2						
3						
4						

Collect results from the different pairs of students and assemble them into the table. If there are still gaps, then allocate different members of the class with squares to draw until you have about 5 rows and 4 columns.

Ask students if they have figured out a number pattern for each y-value. Record these also on the board.

You may end up with a table like this:

x / y	0	1	2	3	4	Comments
0	0	1	4	9	16	They are all square numbers, x^2
1	1	2	5	10	17	They are all one more than square numbers, $x^2 + 1$
2	4	5	8	13	20	They are all four more than x^2 numbers, $x^2 + 4$
3	9	10	13	18	25	They are all nine more than x^2 numbers, $x^2 + 9$
4	16	17	20	25	32	They are all sixteen more than x^2 numbers, $x^2 + 16$

Encourage students to generalize and predict new rows of the table:

Can you tell me what the 5th row will look like? [25, 26, 29, 34, … x^2+5^2]
How can you tell?

Some students will notice that the differences between terms vertically are successive odd numbers. Others may continue the pattern in the comments.

Can you tell me what the 10th row will look like? [100, 101, 104, 109, … x^2+10^2]

Can you tell me what what is the area of a (x,y) square? [x^2+y^2]

Allow students a few minutes to think about this problem individually, and then they should discuss the question with a partner.

Students may provide answers for a specific value of x, for example when $x = 0$ the area is y^2, or when $x = 1$ the area is $1+ y^2$, or when $x = 2$ the area is $4 + y^2$ and so on.

Encourage students to generalize further to a (x, y) square.

This exploration suggests that the area of the square will always be $x^2 + y^2$.

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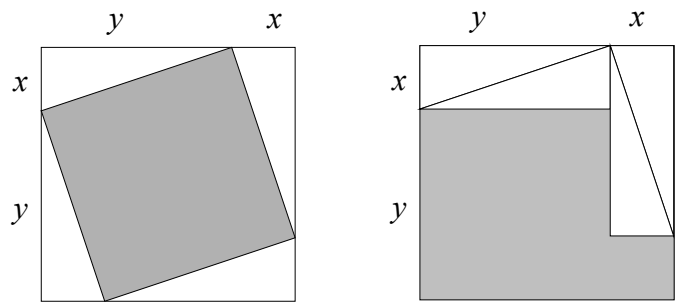
Whole class discussion: Proving the Pythagorean theorem (15 minutes)

Introduce the task:

We think that the area of the gray square is $x^2 + y^2$. But can we prove it?

Project the following two figures on the board using the PowerPoint slides and the data projector.



Ask the following questions in turn:

Can you explain to me why the two gray areas are equal?

What is the area of the second gray area? [$x^2 + y^2$] How can you see this?

So what is the area of the first gray area? [$x^2 + y^2$] Why?

This works for any right triangle with sides x and y.

What is the area of a (50, 100) tilted square?

Ask students to complete the sheet *Proving the Pythagorean Theorem* individually, to summarize the discussion.

Improving individual solutions to the assessment task (10 minutes)

In the next lesson, give students their response to the original assessment task *Square Areas*, as well as a copy of the task *Square Areas (revisited.)*

Look at your original responses and think about what you have learned since you did this.

Make some notes on what you have learned during the lesson.

Now have a go at the second sheet: Square Areas (revisited). Can you use what you have learned to answer these questions?

If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only the questions they think are appropriate to their own work.

If you are short of time, then you could set this task in the next lesson or for homework.

Suggestion for an extension activity

Is it possible to construct a square on the dotted paper with an area of 43?
Can you prove this?

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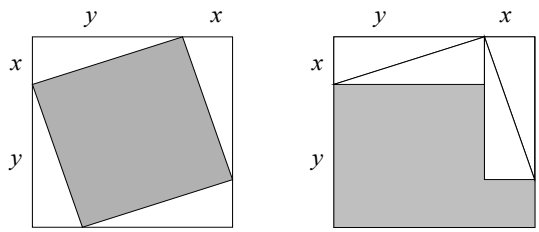
Solutions

Square Areas

1. The shaded area is given by $3^2 + 9^2 = 9 + 25 = 34$ square units.
2. The area of a (3, 7) square is $3^2 + 7^2 = 9 + 49 = 58$ square units.
3. The area of a (*3*, y) square is $3^2 + y^2 = 9 + y^2$ square units.

   The method used to deduce these answers may be similar to Jason's, Kate's or Simon's methods used in the lesson.

Proving the Pythagorean theorem



1. The area of the whole square is equal in both diagrams. The shaded area is this whole square minus four triangles in both diagrams. Thus, we are taking away equal quantities from equal quantities. So the shaded areas are equal.

2. The shaded area in the second diagram is easily seen as $x^2 + y^2$.

3. The hypotenuse of the right triangle with legs x and y is therefore $\sqrt{x^2 + y^2}$.

Square Areas (revisited)

1. The shaded area is given by $7^2 + 6^2 = 49 + 36 = 85$ square units.
2. The area of a (7, 5) square is $7^2 + 5^2 = 49 + 25 = 74$ square units.
3. The area of a (7, y) square is $7^2 + y^2 = 49 + y^2$ square units.

   The method used to deduce these answers may be similar to Jason's, Kate's or Simon's methods used in the lesson or students may use the Pythagorean theorem.

Further notes on possible and impossible areas

It is possible to conjecture some patterns in these sequences. All square numbers leave a remainder of 0 or 1 when divided by 4. So sums of two square numbers (our areas) must leave a remainder of 0, 1, or 2 when divided by 4. This means that if a number takes the form $4n+3$ then it cannot be a tilted area. So this tells us that 3, 7, 11, 15, … cannot be tilted areas. But the list includes more areas than that.

Characterizing these sequences fully is not easy and it took one of the greatest mathematicians to do it. We are not suggesting that students are capable of this! However, for the interested:

It is interesting to start by listing just the prime numbers in each list:

These *can* be made: 2, 5, 13, 17, 29, …

These *cannot* be made: 3, 7, 11, 19, 23, …

This indicates that, apart from 2, primes that are 1 more than multiples of 4 can be made, while primes that are 3 more than multiples of 4 cannot be made.

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- Fermat proved that a prime number p can be expressed as the sum of two squares if and only if p can be written as $4n+1$ where n is an integer.
- If two integers, x and y, can each be written as the sum of two squares, then their product, xy can be written as the sum of two squares. This just (!) involves some algebraic manipulation (easiest with complex numbers) to show that:
  $(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2$.
- Combining these results, here is the theorem that completely solves the problem: A number n is expressible as a sum of two squares if and only if in the prime factorization of n, every prime of the form $(4k+3)$ occurs an even number of times!

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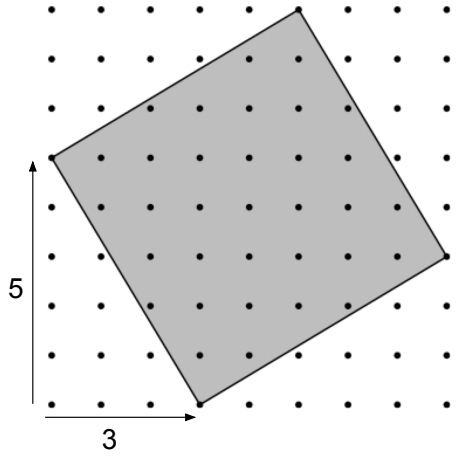
Square Areas

The dots on the grid are all one unit apart.

1. The square shown here can be described as a (3,5) square. Find its area.

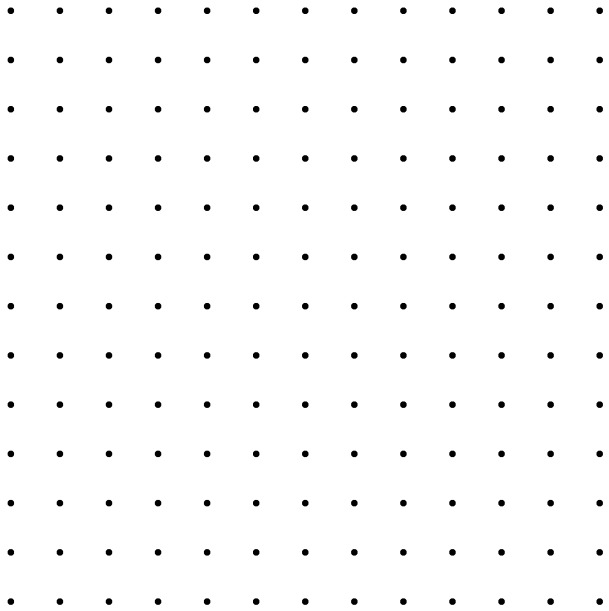
   Show all your reasoning



2. Draw a (3,7) square.

   Find its area.

   Show all your reasoning.



3. Sketch a (3, y) square.

   Find its area in terms of y.

   Show all your reasoning.

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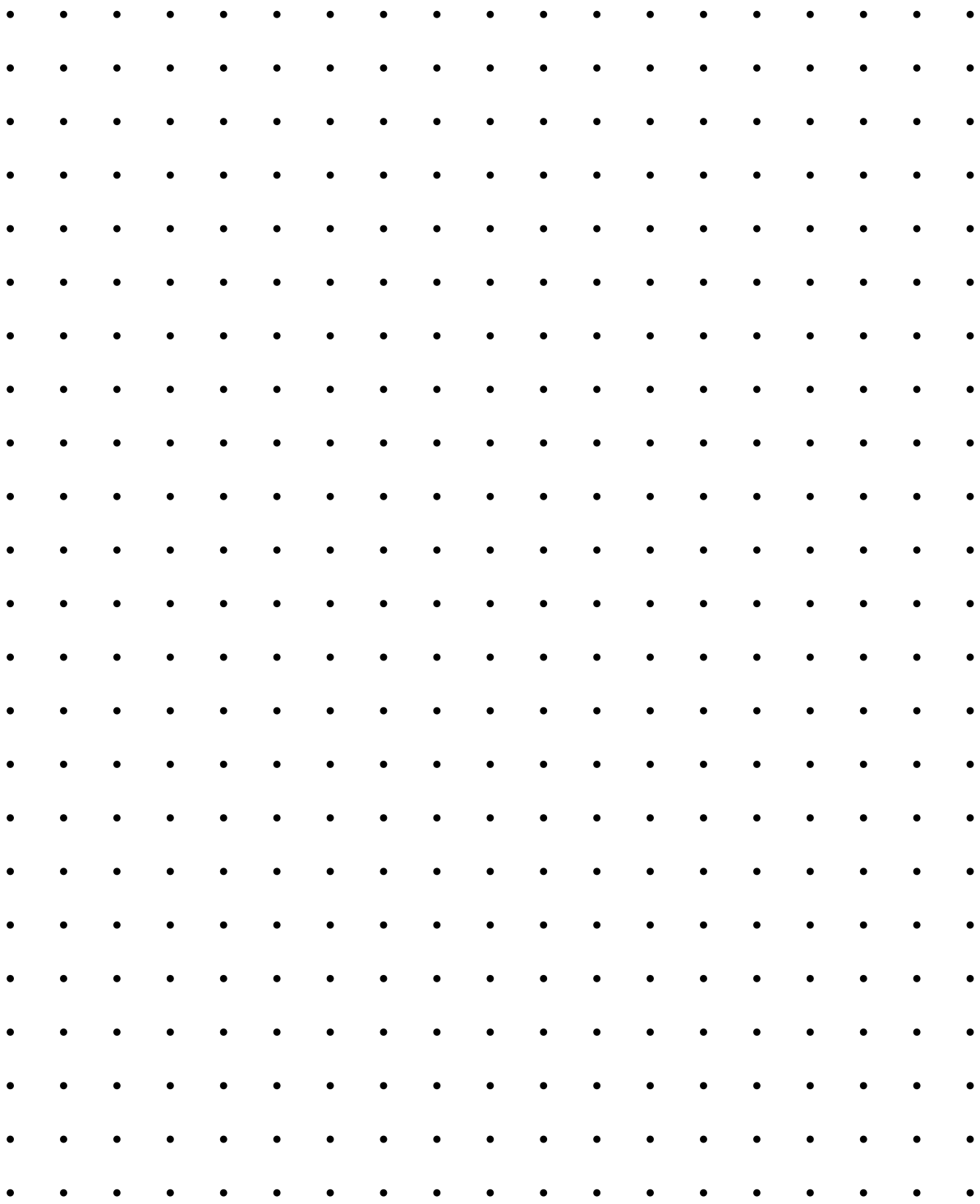
Dotted Grid Paper



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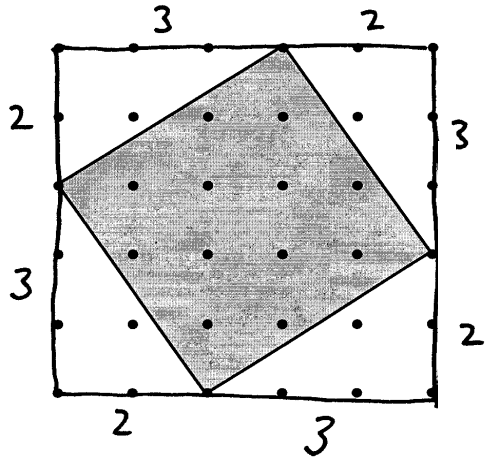
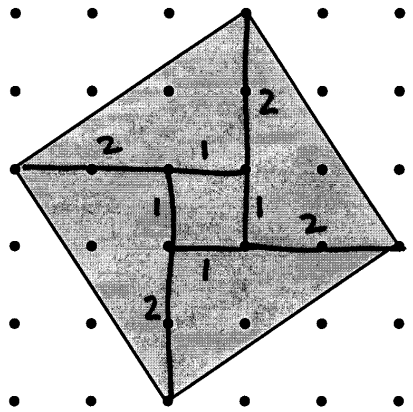
Some Different Approaches



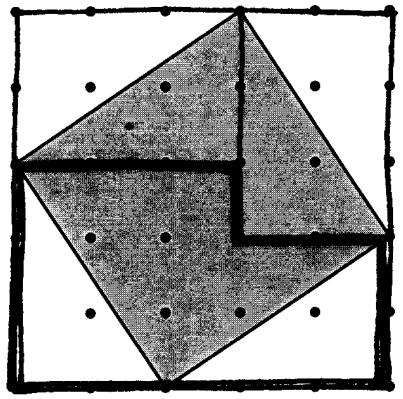
Jason

I drew a square all round the tilted square. I then took away the area of the four right triangles.



Kate

I divided the tilted square into 4 triangles and a little square inside.



Simon

I found that the area inside the bold line is the same area as the tilted square and used that.

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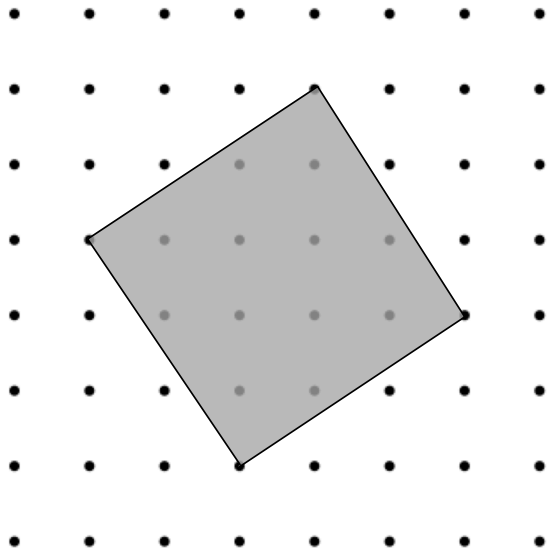
Tilted Squares

1. Find the area of the (2, 3) square.
   Show all your reasoning.

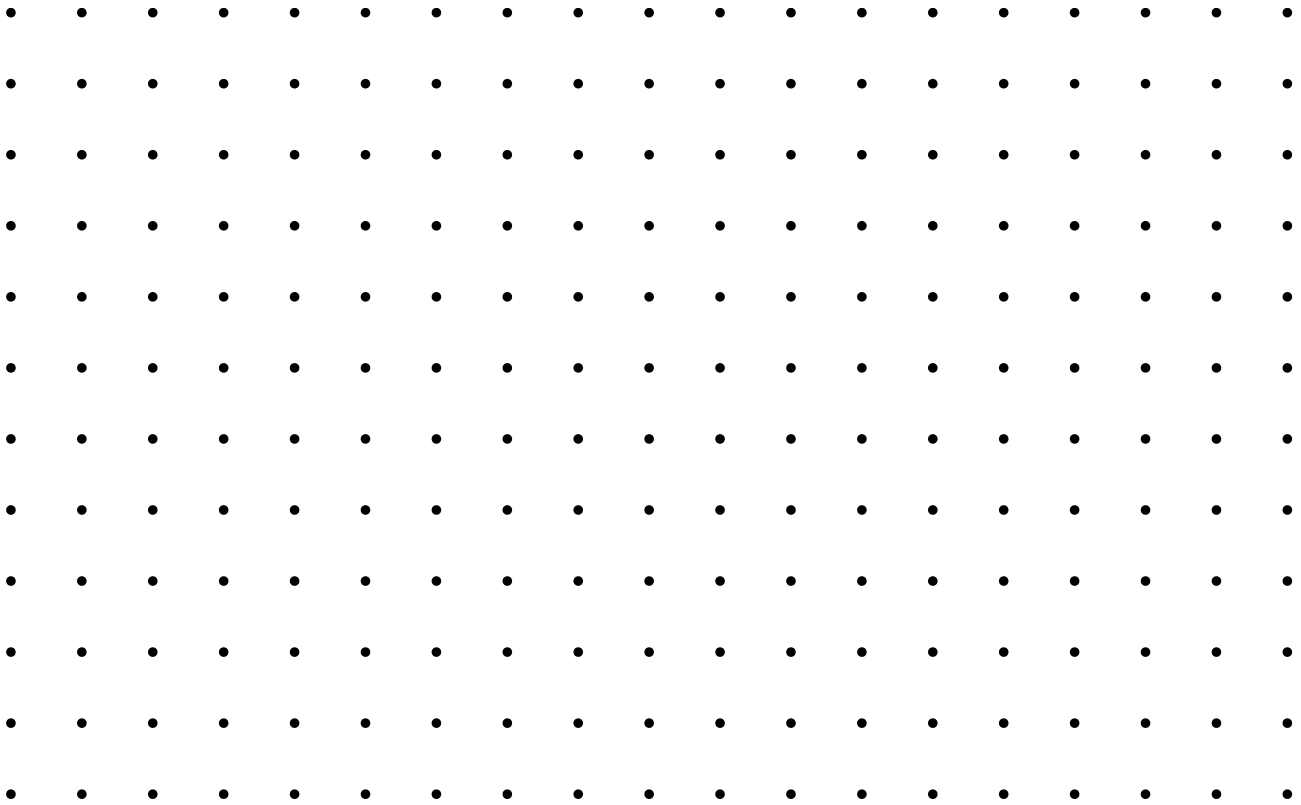
A (2,3) Square



2. In pairs, see if you can find a quick way to figure out the areas of tilted squares.

   What areas can you make by drawing squares on a grid?

   Remember that you must always join dots to make the squares!



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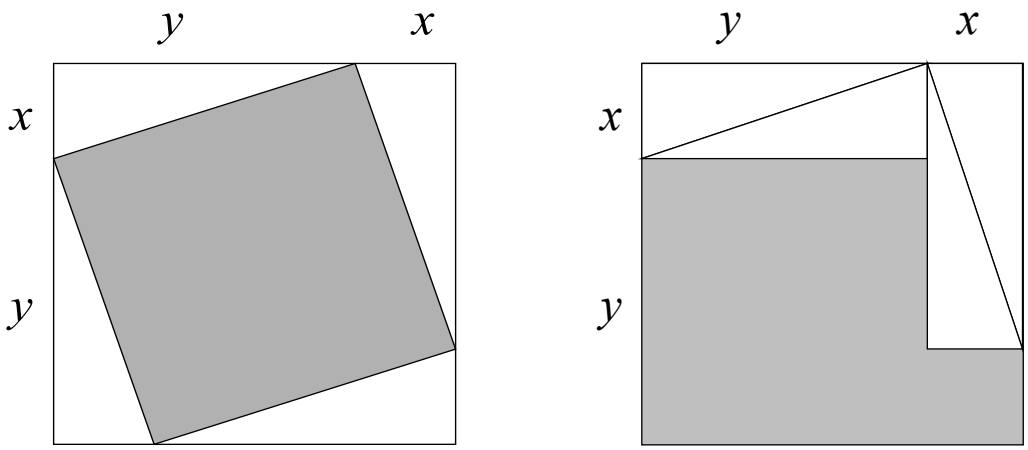
Proving the Pythagorean Theorem



Use what you have learned from the tilted squares problem to do the following:

1. Explain clearly and carefully how you know that the two shaded areas are equal.

2. Write down the shaded areas in terms of the lengths x and y.

3. If the two shorter sides of a right triangle have lengths x and y, what is the length of the longest side? (This is called the hypotenuse).

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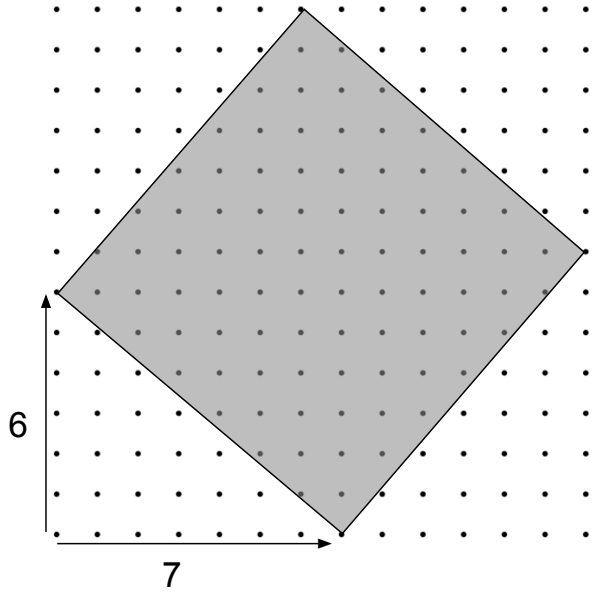
Square Areas (revisited)

The dots on the grid are all one unit apart.

1. The square shown here can be described as a (7,6) square. Find its area.

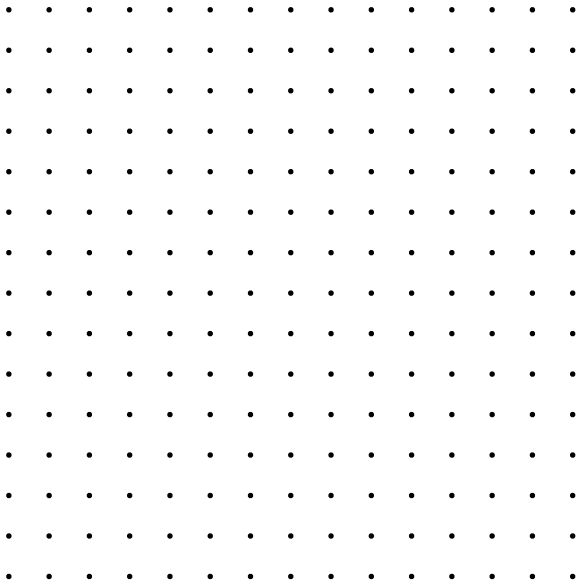
   Show all your reasoning



2. Draw a (7,5) square.

   Find its area.

   Show all your reasoning.



3. Sketch a $(7, y)$ square.

   Find its area in terms of y.

   Show all your reasoning.

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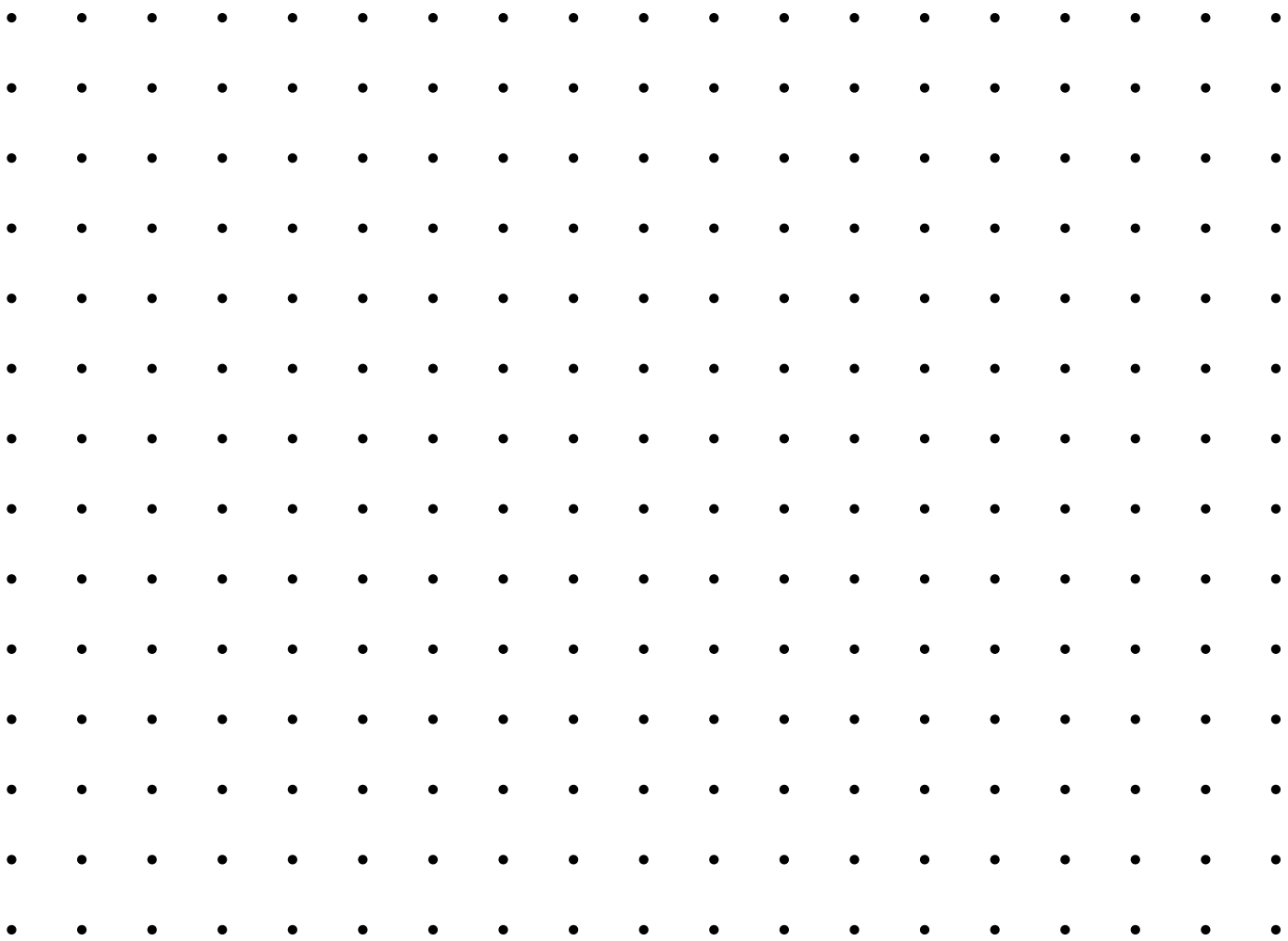
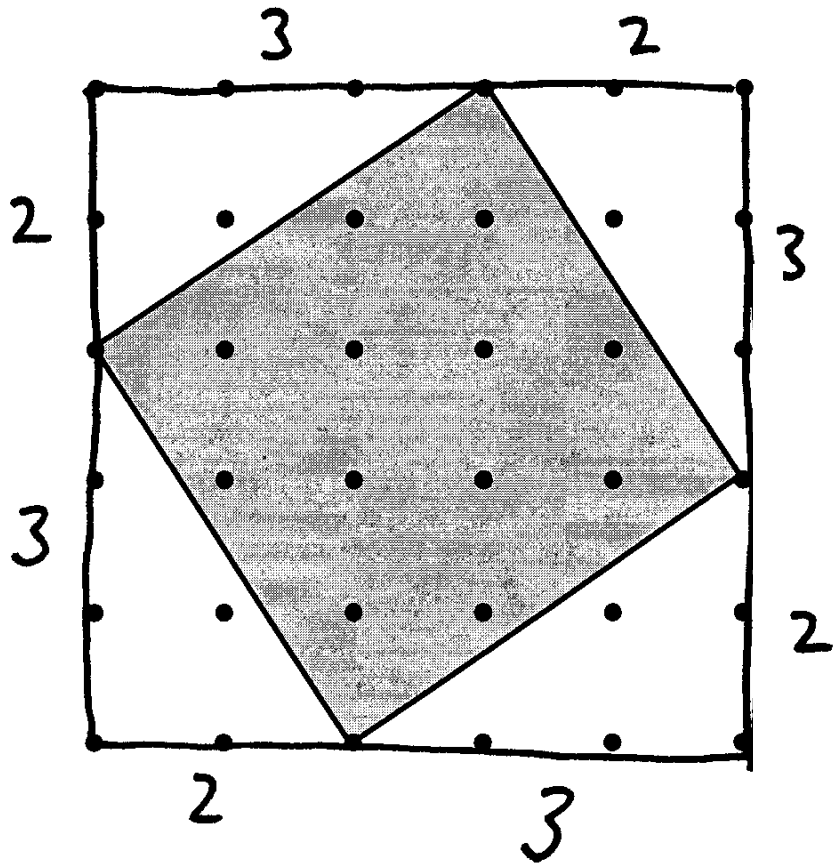


Jason's Method



"I drew a square all round the tilted square.
I then took away the area of the four right triangles."

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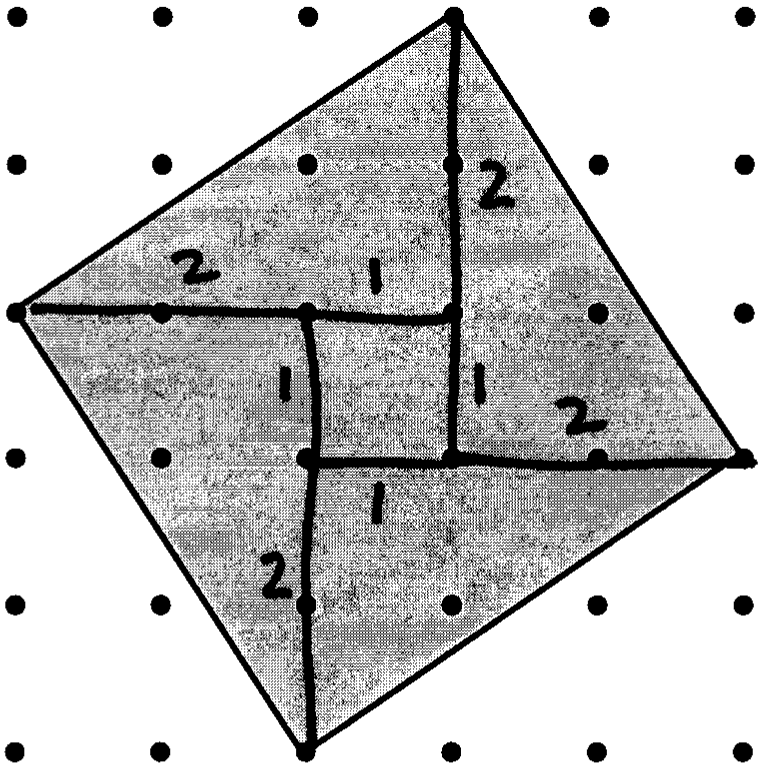
Kate's Method



"I divided the tilted squares into four right triangles and little squares inside."

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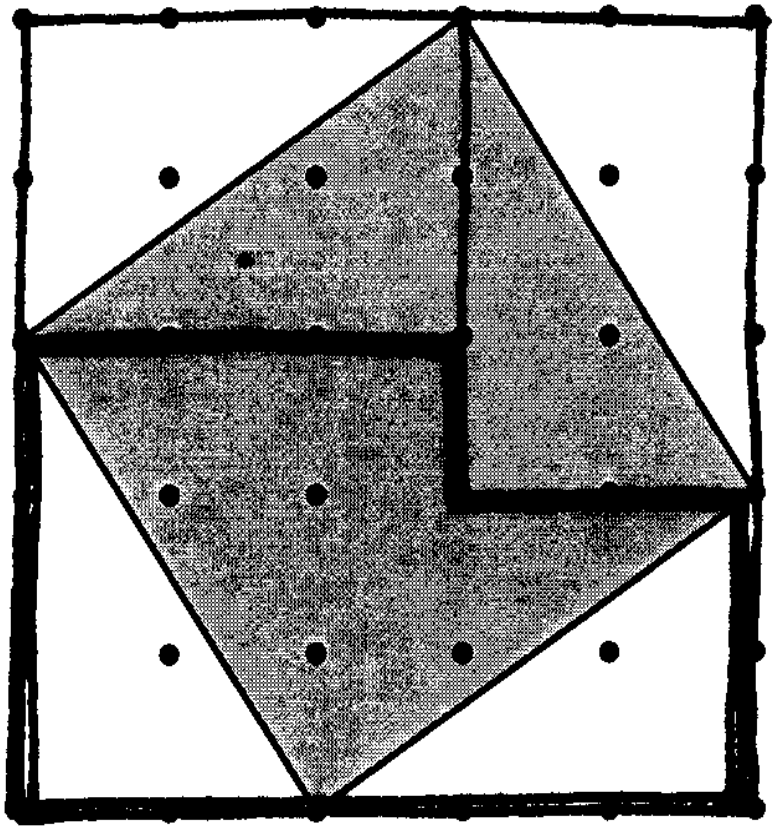
Simon's Method



"I found the area inside the bold line is the same area as the tilted square and used that."

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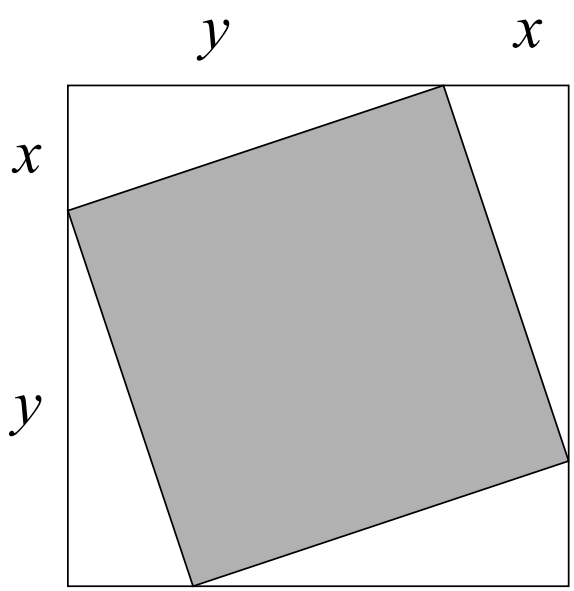
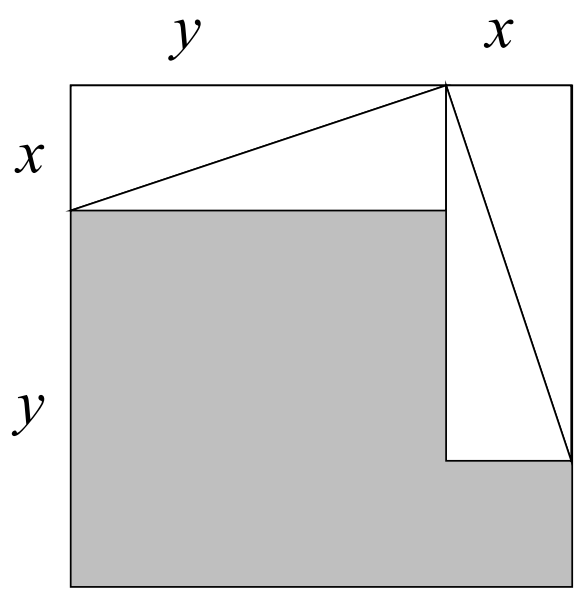
y \ x	0	1	2	3	4	Comments
0						
1						
2						
3						
4						

What is the gray area in each case?





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